

Extended Delta and Nabla Operators and its Sums

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Abstract

This paper aims to the generalized $\alpha(q)$ - difference operator and its sums. By equating the closed and summation forms of inverse operators, we derive novel summation formulae. These advancements contribute to a deeper understanding of the calculus framework and its applications across various mathematical and scientific fields. This results can be applied to improve discrete dynamic systems, enhance computational techniques and provide new insights into the behaviour of discrete functions.

Key words: Delta Operator, Delta ℓ - Operator, Nabla Operator, Nabla ℓ - Operator

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1 Introduction

A difference equation is an equation that contains sequence differences. There are various types of difference equations namely ordinary, delay, advanced, neutral, quasilinear, half linear, etc. These equations occur in numerous settings and forms, both in mathematics itself and its applications to Biology, Computer Science, Digital Signal Processing, Economics, Statistics and other fields. The theory of difference equations, the methods used and their wide applications have advanced beyond their adolescent stage to occupy a central position in applicable analysis. In fact, in the last 15 years, the proliferation of the subject has been witnessed by hundreds of research articles, several monographs, many international conferences and numerous special sessions.

Difference Equation

In numerical integration of differential equations a standard approach is to replace it by a suitable difference equation whose solution can be obtained

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in a stable manner and without troubles from round-off errors. However, the qualitative properties of solutions of the difference equations are quite different from the solutions of the corresponding differential equations. Solutions of several well known difference equations like Clairaut's, Euler's, Riccati's, Bernoulli's, Verhulst's, Duffing's, Mathieu's and Volterra's difference equations preserve most of the properties of the corresponding differential equations.

We solve a difference equation by finding a sequence that satisfies the equation and we call that sequence a solution of the equation.

Defining a difference operator which is typically used in vector calculus for operator like gradient or divergence. In this discrete difference operator, often used in the content of time-series analysis or difference equations. This type of operator computes the difference between the values of the function at the previous time step. Mathematically, this is called forward difference. So, you can interpret this as: $\nabla F(t) = f(t) - f(t - 1)$. This operator is widely used in various fields such as numerical analysis, signal processing, and economics. It gives a way to capture the rate of change successive time points in a discrete setting.

2 Delta Operator

Definition 2.1 Let f be a real valued function and $\zeta \neq 0$. The Delta operator on f is defined as

$$\Delta f(\zeta) = f(\zeta + 1) - f(\zeta). \quad (1)$$

The inverse of Delta operator on f is defined by, if there exists $g(\zeta)$ such that

$$\Delta g(\zeta) = f(\zeta) \Leftrightarrow g(\zeta) + c = \Delta^{-1} f(\zeta), \quad (2)$$

where c is an arbitrary constant.

Theorem 2.2 If f be a real valued function and $\Delta^{-1} f(\zeta)$

$$\sum_{r=0}^m f(\zeta + m - r) = \Delta^{-1} f(\zeta + m + 1) - \Delta^{-1} f(\zeta), \forall m \in \mathbb{Z}^+. \quad (3)$$

Proof: From equation (1), $f(\zeta) = f(\zeta + 1) - f(\zeta)$.

Consider $\Delta^{-1} f(\zeta) = g(\zeta)$, then $f(\zeta) = \Delta g(\zeta)$

Since $f(\zeta) = g(\zeta + 1) - g(\zeta)$.

$$f(\zeta) + g(\zeta) = g(\zeta + 1). \tag{4}$$

Replace ζ by $\zeta + 1$ in equation (4),

$$f(\zeta + 1) + g(\zeta + 1) = g(\zeta + 2)$$

Substute equation (4),

$$f(\zeta + 1) + f(\zeta) + g(\zeta) = g(\zeta + 2).$$

Again replace ζ by $\zeta + 1$,

$$f(\zeta + 2) + f(\zeta + 1) + g(\zeta + 1) = g(\zeta + 3)$$

Substitute equation (4),

$$f(\zeta + 2) + f(\zeta + 1) + f(\zeta) + g(\zeta) = g(\zeta + 3).$$

⋮

By induction;

$$\begin{aligned} f(\zeta + m) + f(\zeta + m - 1) + \dots + f(\zeta) + g(\zeta) &= g(\zeta + m + 1) \\ f(\zeta) + f(\zeta + 1) + \dots + f(\zeta + m) &= \Delta^{-1}f(\zeta + m + 1) - \Delta^{-1}f(\zeta) \end{aligned}$$

From the above result, we get the equation (3).

Example 2.3 Applying $f(\zeta) = \zeta^{(3)}$, $\zeta = 2$ and $m = 3$ in equation (3), then we have

$$\text{LHS} = (\zeta)^{(3)} + (\zeta + 1)^{(3)} + (\zeta + 2)^{(3)} + (\zeta + 3)^{(3)} = (2)^{(3)} + (2 + 1)^{(3)} + (2 + 2)^{(3)} + (2 + 3)^{(3)}$$

$$= (2)^{(3)} + (3)^{(3)} + (4)^{(3)} + (5)^{(3)} = 0 + 6 + 24 + 60 = 90.$$

We know that, $\Delta^{-1}\zeta^{(3)} = \frac{\zeta^4}{4}$

$$\text{RHS} = \frac{(2 + 3 + 1)^{(4)}}{4} - \frac{2^{(4)}}{4} = \frac{6.5.4.3}{4} - \frac{2.1.0. - 1}{4} = 90.$$

Therefore, LHS=RHS=90.

Theorem 2.4 If f be a real valued function, $\Delta^{-1}f(\zeta)$ and $\Delta^{-2}f(\zeta)$ exists, then

$$\sum_{r=0}^m (r+1)f(\zeta + m - r) = \Delta^{-2}f(\zeta + m + 2) - \Delta^{-2}f(\zeta) - (m+2)\Delta^{-1}f(\zeta), \forall m \in Z^+. \tag{5}$$

Proof: By equation (3), then we have

$$\sum_{r=0}^m f(\zeta + m - r) = \Delta^{-1}f(\zeta + m + 1) - \Delta^{-1}f(\zeta)$$

Applying Δ^{-1} on both sides,

$$\Delta^{-1}f(\zeta) + \Delta^{-1}f(\zeta + 1) + \cdots + \Delta^{-1}f(\zeta + m) = \Delta^{-2}f(\zeta + m + 1) - \Delta^{-2}f(\zeta)$$

Put $m = 3$ in equation (3),

$$f(\zeta) + f(\zeta + 1) + f(\zeta + 2) + f(\zeta + 3) = \Delta^{-1}f(\zeta + 4) - \Delta^{-1}f(\zeta) \quad (6)$$

Put $m = 2$ in equation (3),

$$f(\zeta) + f(\zeta + 1) + f(\zeta + 2) = \Delta^{-1}f(\zeta + 3) - \Delta^{-1}f(\zeta) \quad (7)$$

Put $m = 1$ in equation (3),

$$f(\zeta) + f(\zeta + 1) = \Delta^{-1}f(\zeta + 2) - \Delta^{-1}f(\zeta) \quad (8)$$

Put $m = 0$ in equation (3),

$$f(\zeta) = \Delta^{-1}f(\zeta + 1) - \Delta^{-1}f(\zeta) \quad (9)$$

$$0 = \Delta^{-1}f(\zeta) - \Delta^{-1}f(\zeta) \quad (10)$$

Adding (6) to (10), we get

$$4f(\zeta) + 3f(\zeta + 1) + 2f(\zeta + 2) + f(\zeta + 3) = \Delta^{-2}f(\zeta + 5) - \Delta^{-2}f(\zeta) - 5\Delta^{-1}f(\zeta)$$

In general,

$$\begin{aligned} (m+1)f(\zeta) + mf(\zeta + 1) + (m-1)f(\zeta + 2) + \cdots + (1)f(\zeta + m) \\ = \Delta^{-2}f(\zeta + m + 2) - \Delta^{-2}f(\zeta) - (m+2)\Delta^{-1}f(\zeta) \end{aligned}$$

From the above result, we get the equation (5).

Example 2.5 Applying $f(\zeta) = \zeta^{(3)}$, $\zeta = 2$ and $m=3$ in equation (5), then we have

$$\begin{aligned} \text{LHS} &= (\zeta + 3)^{(3)} + 2(\zeta + 2)^{(3)} + 3(\zeta + 1)^{(3)} + 4(\zeta)^{(3)} \\ &= (2 + 3)^{(3)} + 2(2 + 2)^{(3)} + 3(2 + 1)^{(3)} + 4(2)^{(3)} = 60 + 2(24) + 3(6) + 0 = 126. \end{aligned}$$

We know that, $\Delta^{-1}\zeta^{(3)} = \frac{\zeta^4}{4}$ and $\Delta^{-2}\zeta^{(3)} = \frac{\zeta^5}{5.4}$

$$\begin{aligned} \text{RHS} &= \frac{(2+3+2)^{(5)}}{5.4} - \frac{(2)^{(5)}}{5.4} - (3+2)\frac{(2)^{(4)}}{4} = \frac{(7)^{(5)}}{5.4} - \frac{(2)^{(5)}}{5.4} - (5)\frac{(2)^{(4)}}{4} \\ &= \frac{(7.6.5.4.3)}{5.4} - 0 - (5)\frac{(2.1.0)^{(3)}}{3} = 126. \end{aligned}$$

Therefore, LHS=RHS=126.

Theorem 2.6 If f be a real valued function and $\Delta^{-1}f(\zeta)$, $\Delta^{-2}f(\zeta)$ and $\Delta^{-3}f(\zeta)$ exist, then

$$\begin{aligned} \sum_{r=0}^m \frac{(r+2)^{(2)}}{2} f(\zeta + m - r) &= \Delta^{-3}f(\zeta + m + 3) - \Delta^{-3}f(\zeta) - (m+3)\Delta^{-2}f(\zeta) \\ &\quad - \frac{(m+3)^{(2)}}{2} \Delta^{-1}f(\zeta), \forall m \in \mathbb{Z}^+. \end{aligned} \quad (11)$$

Proof: By equation (5), we have

$$\sum_{r=0}^m (r+1)f(\zeta + m - r) = \Delta^{-2}f(\zeta + m + 2) - \Delta^{-2}f(\zeta) - (m+2)\Delta^{-1}f(\zeta)$$

Put $m = 4$ in equation (5),

$$f(\zeta + 4) + 2f(\zeta + 3) + 3f(\zeta + 2) + 4f(\zeta + 1) + 5f(\zeta) = \Delta^{-2}f(\zeta + 6) - \Delta^{-2}f(\zeta) - (6)\Delta^{-1}f(\zeta) \quad (12)$$

Put $m = 3$ in equation (5),

$$f(\zeta + 3) + 2f(\zeta + 2) + 3f(\zeta + 1) + 4f(\zeta) = \Delta^{-2}f(\zeta + 5) - \Delta^{-2}f(\zeta) - (5)\Delta^{-1}f(\zeta) \quad (13)$$

Put $m = 2$ in equation (5),

$$f(\zeta + 2) + 2f(\zeta + 1) + 3f(\zeta) = \Delta^{-2}f(\zeta + 4) - \Delta^{-2}f(\zeta) - (4)\Delta^{-1}f(\zeta) \quad (14)$$

Put $m = 1$ in equation (5),

$$f(\zeta + 1) + 2f(\zeta) = \Delta^{-2}f(\zeta + 3) - \Delta^{-2}f(\zeta) - (3)\Delta^{-1}f(\zeta) \quad (15)$$

Put $m = 0$ in equation (5),

$$f(\zeta) = \Delta^{-2}f(\zeta + 2) - \Delta^{-2}f(\zeta) - (2)\Delta^{-1}f(\zeta) \tag{16}$$

$$0 = \Delta^{-2}f(\zeta + 1) - \Delta^{-2}f(\zeta) - (1)\Delta^{-1}f(\zeta) \tag{17}$$

$$0 = \Delta^{-2}f(\zeta) - \Delta^{-2}f(\zeta) - (0)\Delta^{-1}f(\zeta) \tag{18}$$

Adding (12) to (18), we get

In general,

$$\begin{aligned} & f(\zeta + 4) + (1 + 2)f(\zeta + 3) + (1 + 2 + 3)f(\zeta + 2) + (1 + 2 + 3 + 4)f(\zeta + 1) \\ & \qquad \qquad \qquad + (1 + 2 + 3 + 4 + 5)f(\zeta) \\ & = \Delta^{-3}f(\zeta + m + 3) - \Delta^{-3}f(\zeta) - (m + 3)\Delta^{-2}f(\zeta) - \frac{(7)^{(2)}}{2}\Delta^{-1}f(\zeta) \end{aligned}$$

From the above result, we get the equation (11).

Example 2.7 Applying $f(\zeta) = \zeta^{(3)}$, $\zeta = 2$ and $m = 3$ in equation (11), then we have

$$\begin{aligned} \text{LHS} &= \sum_{r=0}^3 \frac{(r+2)^{(2)}}{2} f(\zeta + 3 - r) \\ &= \frac{(2)^{(2)}}{2} (2+3)^{(3)} + \frac{(3)^{(2)}}{2} (2+2)^{(3)} + \frac{(4)^{(2)}}{2} (2+1)^{(3)} + \frac{(5)^{(2)}}{2} (2)^{(3)} \\ &= 5^{(3)} + 3(4)^{(3)} + 6(3)^{(3)} + 10(2)^{(3)} = 60 + 3(24) + 6(6) + 10(0) = 168. \end{aligned}$$

We know that, $\Delta^{-1}\zeta^{(3)} = \frac{\zeta^4}{4}$, $\Delta^{-2}\zeta^{(3)} = \frac{\zeta^5}{5.4}$ and $\Delta^{-3}\zeta^{(3)} = \frac{\zeta^6}{6.5.4}$

$$\text{RHS} = \frac{(2+3+3)^{(6)}}{6.5.4} - \frac{(2)^{(6)}}{6.5.4} - 6 \frac{(2)^{(5)}}{5.4} - \frac{(6)^{(2)}(2)^{(4)}}{2 \cdot 4} = \frac{(8.7.6.5.4.3)}{6.5.4} - 0 - 0 - 0 = 168.$$

Therefore, LHS=RHS=168.

Theorem 2.8 If f be a real valued function and $\Delta^{-1}f(\zeta)$, $\Delta^{-2}f(\zeta)$, $\Delta^{-3}f(\zeta)$ and $\Delta^{-4}f(\zeta)$ exists, then

$$\begin{aligned} \sum_{r=0}^m \frac{(r+3)^{(3)}}{3^{(2)}} f(\zeta + m - r) &= \Delta^{-4}f(\zeta + m + 4) - \Delta^{-4}f(\zeta) - (m + 4)\Delta^{-3}f(\zeta) \\ &\quad - \frac{(m+4)^{(2)}}{2}\Delta^{-2}f(\zeta) - \frac{(m+4)^{(3)}}{3^{(2)}}\Delta^{-1}f(\zeta), \forall m \in Z^+. \end{aligned} \tag{19}$$

Proof: By equation (11), we have

$$\sum_{r=0}^m \frac{(r+2)^{(2)}}{2} f(\zeta+m-r) = \Delta^{-3} f(\zeta+m+3) - \Delta^{-3} f(\zeta) - (m+3)\Delta^{-2} f(\zeta) - \frac{(m+3)^{(2)}}{2} \Delta^{-1} f(\zeta)$$

Put $m = 4$ in equation (11),

$$\begin{aligned} \sum_{r=0}^3 \frac{(r+2)^{(2)}}{2} f(\zeta+4-r) &= \Delta^{-3} f(\zeta+4+3) - \Delta^{-3} f(\zeta) - (4+3)\Delta^{-2} f(\zeta) \\ &\quad - \frac{(4+3)^{(2)}}{2} \Delta^{-1} f(\zeta), \\ \frac{(2)^{(2)}}{2} f(\zeta+4) + \frac{(3)^{(2)}}{2} f(\zeta+3) + \frac{(4)^{(2)}}{2} f(\zeta+2) + \frac{(5)^{(2)}}{2} f(\zeta+1) + \frac{(6)^{(2)}}{2} f(\zeta) \\ &= \Delta^{-3} f(\zeta+6) - \Delta^{-3} f(\zeta) - (6)\Delta^{-2} f(\zeta) - \frac{(6)^{(2)}}{2} \Delta^{-1} f(\zeta) \\ f(\zeta+4) + (3)f(\zeta+3) + (6)f(\zeta+2) + (10)f(\zeta+1) + (15)f(\zeta) \\ &= \Delta^{-3} f(\zeta+7) - \Delta^{-3} f(\zeta) - (7)\Delta^{-2} f(\zeta) - \frac{(7)^{(2)}}{2} \Delta^{-1} f(\zeta) \end{aligned} \quad (20)$$

Put $m = 3$ in equation (11),

$$\begin{aligned} \sum_{r=0}^3 \frac{(r+2)^{(2)}}{2} f(\zeta+3-r) &= \Delta^{-3} f(\zeta+3+3) - \Delta^{-3} f(\zeta) - (3+3)\Delta^{-2} f(\zeta) \\ &\quad - \frac{(3+3)^{(2)}}{2} \Delta^{-1} f(\zeta), \\ \frac{(2)^{(2)}}{2} f(\zeta+3) + \frac{(3)^{(2)}}{2} f(\zeta+2) + \frac{(4)^{(2)}}{2} f(\zeta+1) + \frac{(5)^{(2)}}{2} f(\zeta) \\ &= \Delta^{-3} f(\zeta+6) - \Delta^{-3} f(\zeta) - (6)\Delta^{-2} f(\zeta) - \frac{(6)^{(2)}}{2} \Delta^{-1} f(\zeta) \\ f(\zeta+3) + (3)f(\zeta+2) + (6)f(\zeta+1) + (10)f(\zeta) \\ &= \Delta^{-3} f(\zeta+6) - \Delta^{-3} f(\zeta) - (6)\Delta^{-2} f(\zeta) - \frac{(6)^{(2)}}{2} \Delta^{-1} f(\zeta) \end{aligned} \quad (21)$$

Put $m = 2$ in equation (11),

$$\begin{aligned} \sum_{r=0}^2 \frac{(r+2)^{(2)}}{2} f(\zeta+2-r) &= \Delta^{-3} f(\zeta+2+3) - \Delta^{-3} f(\zeta) - (2+3)\Delta^{-2} f(\zeta) \\ &\quad - \frac{(2+3)^{(2)}}{2} \Delta^{-1} f(\zeta), \\ \frac{(2)^{(2)}}{2} f(\zeta+2) + \frac{(3)^{(2)}}{2} f(\zeta+1) + \frac{(4)^{(2)}}{2} f(\zeta) \end{aligned}$$

$$= \Delta^{-3}f(\zeta + 5) - \Delta^{-3}f(\zeta) - (5)\Delta^{-2}f(\zeta) - \frac{(5)^{(2)}}{2}\Delta^{-1}f(\zeta)$$

$$f(\zeta + 2) + (3)f(\zeta + 1) + (6)f(\zeta) = \Delta^{-3}f(\zeta + 5) - \Delta^{-3}f(\zeta) - (5)\Delta^{-2}f(\zeta) - \frac{(5)^{(2)}}{2}\Delta^{-1}f(\zeta) \quad (22)$$

Put $m = 1$ in equation (11),

$$\sum_{r=0}^1 \frac{(r+2)^{(2)}}{2} f(\zeta + 1 - r) = \Delta^{-3}f(\zeta + 1 + 3) - \Delta^{-3}f(\zeta) - (1+3)\Delta^{-2}f(\zeta) - \frac{(1+3)^{(2)}}{2}\Delta^{-1}f(\zeta),$$

$$\frac{(2)^{(2)}}{2}f(\zeta + 1) + \frac{(3)^{(2)}}{2}f(\zeta) = \Delta^{-3}f(\zeta + 4) - \Delta^{-3}f(\zeta) - (4)\Delta^{-2}f(\zeta) - \frac{(4)^{(2)}}{2}\Delta^{-1}f(\zeta)$$

$$f(\zeta + 1) + (3)f(\zeta) = \Delta^{-3}f(\zeta + 4) - \Delta^{-3}f(\zeta) - (4)\Delta^{-2}f(\zeta) - \frac{(4)^{(2)}}{2}\Delta^{-1}f(\zeta) \quad (23)$$

Put $m = 0$ in equation (11)

$$\frac{(2)^{(2)}}{2}f(\zeta) = \Delta^{-3}f(\zeta + 3) - \Delta^{-3}f(\zeta) - (3)\Delta^{-2}f(\zeta) - \frac{(3)^{(2)}}{2}\Delta^{-1}f(\zeta)$$

$$f(\zeta) = \Delta^{-3}f(\zeta + 3) - \Delta^{-3}f(\zeta) - (3)\Delta^{-2}f(\zeta) - \frac{(3)^{(2)}}{2}\Delta^{-1}f(\zeta) \quad (24)$$

$$0 = \Delta^{-3}f(\zeta + 2) - \Delta^{-3}f(\zeta) - (2)\Delta^{-2}f(\zeta) - \frac{(2)^{(2)}}{2}\Delta^{-1}f(\zeta) \quad (25)$$

$$0 = \Delta^{-3}f(\zeta + 1) - \Delta^{-3}f(\zeta) - (1)\Delta^{-2}f(\zeta) - \frac{(1)^{(2)}}{2}\Delta^{-1}f(\zeta) \quad (26)$$

$$0 = \Delta^{-3}f(\zeta) - \Delta^{-3}f(\zeta) \quad (27)$$

Adding equations (20) to (27), we get

$$\begin{aligned} & f(\zeta + 4) + (1+3)f(\zeta + 3) + (1+3+6)f(\zeta + 2) + (1+3+6+10)f(\zeta + 1) \\ & \quad + (1+3+6+10+15)f(\zeta) \\ & \quad = \\ & \Delta^{-4}f(\zeta + m + 4) - \Delta^{-4}f(\zeta) - (m+4)\Delta^{-3}f(\zeta) - \frac{(m+4)^{(2)}}{2}\Delta^{-2}f(\zeta) \\ & \quad - (1+3+6+10+15+21)\Delta^{-1}f(\zeta) \end{aligned}$$

In general,

$$f(\zeta + 4) + (4)f(\zeta + 3) + (10)f(\zeta + 2) + (20)f(\zeta + 1) + (35)f(\zeta)$$

$$= \Delta^{-4}f(\zeta + m + 4) - \Delta^{-4}f(\zeta) - (m + 4)\Delta^{-3}f(\zeta) - \frac{(m + 4)^{(2)}}{2}\Delta^{-2}f(\zeta) - (56)\Delta^{-1}f(\zeta)$$

From the above result, we get the equation (19).

Example 2.9 Applying $f(\zeta) = \zeta^{(3)}$, $\zeta = 2$ and $m=3$ in equation (19), then we have

$$\begin{aligned} \text{LHS} &= \sum_{r=0}^3 \frac{(r+3)^{(3)}}{3^{(2)}} f(\zeta + 3 - r) \\ &= \frac{(3)^{(3)}}{3^{(2)}} (2+3)^{(3)} + \frac{(4)^{(3)}}{3^{(2)}} (2+2)^{(3)} + \frac{(5)^{(3)}}{3^{(2)}} (2+1)^{(3)} + \frac{(6)^{(3)}}{3^{(2)}} (2)^{(3)} \\ &= 5 + 4(4)^{(3)} + 10(3)^{(3)} + 20(2)^{(3)} = 60 + 4(24) + 10(6) + 20(0) = 216. \end{aligned}$$

We know that, $\Delta^{-1}\zeta^{(3)} = \frac{\zeta^4}{4}$, $\Delta^{-2}\zeta^{(3)} = \frac{\zeta^5}{5.4}$, $\Delta^{-3}\zeta^{(3)} = \frac{\zeta^6}{6.5.4}$ and $\Delta^{-4}\zeta^{(3)} = \frac{\zeta^7}{7.6.5.4}$

$$\begin{aligned} \text{RHS} &= \frac{(2+3+4)^{(7)}}{7^{(4)}} - \frac{(2)^{(7)}}{7^{(4)}} - 7 \frac{(2)^{(6)}}{6^{(3)}} - \frac{(7)^{(2)}(2)^{(5)}}{2 \cdot 3^{(2)}} + \frac{7^{(3)} 2^{(4)}}{3^{(2)} \cdot 4} \\ &= \frac{(9)^{(7)}}{7^{(4)}} - 0 - 0 - 0 - 0 = \frac{9.8.7.6.5.4.3}{7.6.5.4} = 216. \end{aligned}$$

Therefore, LHS=RHS=216.

Theorem 2.10 If f be a real valued function and $\Delta^{-r}f(\zeta)$ where $r=1,2,\dots,n$ exists, then

$$\begin{aligned} \sum_{r=0}^m \frac{(r+n-1)^{(n-1)}}{(n-1)!} f(\zeta + m - r) &= \Delta^{-n}f(\zeta + m + n) - \Delta^{-n}f(\zeta) - \frac{(m+n)^{(1)}}{1!}\Delta^{-(n-1)}f(\zeta) \\ &\quad - \frac{(m+n)^{(2)}}{2!}\Delta^{-(n-2)}f(\zeta) \\ &\quad - \frac{(m+n)^{(n-1)}}{(n-1)!}\Delta^{-1}f(\zeta), \forall m \in \mathbb{Z}^+. \end{aligned} \quad (28)$$

Proof: The proof follows from, put $n = 1$ in equation (28),

$$\sum_{r=0}^m f(\zeta + m - r) = \Delta^{-1}f(\zeta + m + 1) - \Delta^{-1}f(\zeta)$$

Put $n = 2$ in equation (28),

$$\sum_{r=0}^m (r+1)f(\zeta + m - r) = \Delta^{-2}f(\zeta + m + 2) - \Delta^{-2}f(\zeta) - (m+2)\Delta^{-1}f(\zeta)$$

Put $n = 3$ in equation (28),

$$\sum_{r=0}^m \frac{(r+2)^{(2)}}{2!} f(\zeta + m - r) = \Delta^{-3}f(\zeta + m + 3) - \Delta^{-3}f(\zeta) - (m+3)\Delta^{-2}f(\zeta)$$

$$- \frac{(m+3)^{(2!)}}{2} \Delta^{-1} f(\zeta)$$

Put $n = 4$ in equation (28),

$$\sum_{r=0}^m \frac{(r+3)^{(3)}}{3!} f(\zeta + m - r) = \Delta^{-4} f(\zeta + m + 4) - \Delta^{-4} f(\zeta) - (m+4) \Delta^{-3} f(\zeta) - \frac{(m+4)^{(2)}}{2!} \Delta^{-2} f(\zeta) - \frac{(m+4)^{(3)}}{3!} \Delta^{-1} f(\zeta).$$

Example 2.11 Put $n = 5$ in equation (28)

$$\begin{aligned} \sum_{r=0}^m \frac{(r+4)^{(4)}}{4!} f(\zeta + m - r) &= \Delta^{-5} f(\zeta + m + 5) - \Delta^{-5} f(\zeta) - \frac{(m+5)^{(1)}}{1!} \Delta^{-4} f(\zeta) \\ &\quad - \frac{(m+5)^{(2)}}{2!} \Delta^{-3} f(\zeta) - \frac{(m+5)^{(3)}}{3!} \Delta^{-2} f(\zeta) - \\ &\quad \frac{(m+5)^{(4)}}{4!} \Delta^{-1} f(\zeta) \end{aligned}$$

Applying $f(\zeta) = \zeta^{(3)}$, $\zeta = 2$ and $m = 3$ in the above equation, we get.

$$\begin{aligned} \text{LHS} &= \sum_{r=0}^3 \frac{(r+4)^{(4)}}{4!} f(\zeta + 3 - r) \\ &= \frac{(4)^{(4)}}{4!} (2+3)^{(3)} + \frac{(5)^{(4)}}{4!} (2+2)^{(3)} + \frac{(6)^{(4)}}{4!} (2+1)^{(3)} + \frac{(7)^{(4)}}{4!} (2)^{(3)} \\ &= 5^{(3)} + 5(4)^{(3)} + 15(3)^{(3)} + 35(2)^{(3)} = 60 + 5(24) + 15(6) + 35(0) = 270 \end{aligned}$$

We know that, $\Delta^{-1} \zeta^{(3)} = \frac{\zeta^4}{4}$, $\Delta^{-2} \zeta^{(3)} = \frac{\zeta^5}{5.4}$, $\Delta^{-3} \zeta^{(3)} = \frac{\zeta^6}{6.5.4}$, $\Delta^{-4} \zeta^{(3)} = \frac{\zeta^7}{7.6.5.4}$
 and $\Delta^{-5} \zeta^{(3)} = \frac{\zeta^7}{8.7.6.5.4}$

$$\begin{aligned} \text{RHS} &= \frac{(2+3+3)^{(8)}}{8.7.6.5.4.3} - \frac{(2)^{(7)}}{7.6.5.4.3} - 8 \frac{(2)^{(6)}}{6.5.4.3} - \frac{(8)^{(2)} (2)^{(5)}}{2! \cdot 60} - \frac{(8)^{(3)} (2)^{(4)}}{3! \cdot 48} - \frac{(8)^{(4)} (2)^{(3)}}{4! \cdot 3} \\ &= \frac{(10.9.8.7.6.5.4)}{7.6.5.4.3} - 0 - 0 - 0 = 270 \end{aligned}$$

Therefore, LHS=RHS=240.

Thus we have obtained Delta operator its inverse and some basic results using these operator.

3 Delta- ℓ Operator

Definition 3.1 Let f be a real valued function and $\zeta \neq 0$. The Delta- ℓ operator on f is defined as

$$\Delta_{\ell}f(\zeta) = f(\zeta + \ell) - f(\zeta). \quad (29)$$

The inverse of Delta- ℓ operator on f is defined by, if there exists $g(\zeta)$ such that

$$\Delta_{\ell}g(\zeta) = f(\zeta) \Leftrightarrow g(\zeta) + c = \Delta_{\ell}^{-1}f(\zeta), \quad (30)$$

where c is an arbitrary constant.

Theorem 3.2 If f be a real valued function and $\Delta_{\ell}^{-1}f(\zeta)$ exists, then

$$\sum_{r=0}^m f(\zeta + (m-r)\ell) = \Delta_{\ell}^{-1}f(\zeta + (m+1)\ell) - \Delta_{\ell}^{-1}f(\zeta), \forall m \in \mathbb{Z}^+. \quad (31)$$

Proof: From equation (29), $f(\zeta) = f(\zeta + \ell) - f(\zeta)$.

Consider $\Delta_{\ell}^{-1}f(\zeta) = g(\zeta)$, then $f(\zeta) = \Delta_{\ell}g(\zeta)$

Since $f(\zeta) = g(\zeta + \ell) - g(\zeta)$,

$$f(\zeta) + g(\zeta) = g(\zeta + \ell) \quad (32)$$

Replace ζ by $\zeta + \ell$ in equation (29),

$$f(\zeta + \ell) + g(\zeta + \ell) = g(\zeta + 2\ell)$$

Substitute equation (32),

$$f(\zeta + \ell) + f(\zeta) + g(\zeta) = g(\zeta + 2\ell)$$

Again replace ζ by $\zeta + \ell$,

$$f(\zeta + 2\ell) + f(\zeta + \ell) + g(\zeta + \ell) = g(\zeta + 3\ell)$$

Substitute equation (32),

$$f(\zeta + 2\ell) + f(\zeta + \ell) + f(\zeta) + g(\zeta) = g(\zeta + 3\ell)$$

⋮

By induction;

$$f(\zeta + m\ell) + f(\zeta + (m-1)\ell) + \cdots + f(\zeta) + g(\zeta) = g(\zeta + (m+1)\ell)$$

$$f(\zeta) + f(\zeta + \ell) + \dots + f(\zeta + m\ell) = \Delta_\ell^{-1} f(\zeta + (m+1)\ell) - \Delta_\ell^{-1} f(\zeta)$$

From the above result, we get the equation (31).

Example 3.3 Applying $f(\zeta) = \zeta_\ell^{(2)}$, $\zeta = 3$, $\ell = 2$ and $m = 3$ in equation (31),

$$\sum_{r=0}^m f(\zeta + (m-r)\ell) = \Delta_\ell^{-1} f(\zeta + (m+1)\ell) - \Delta_\ell^{-1} f(\zeta)$$

$$\text{LHS} = f(\zeta) + f(\zeta + \ell) + f(\zeta + 2\ell) + f(\zeta + 3\ell) = (\zeta)_\ell^{(2)} + (\zeta + \ell)_\ell^{(2)} + (\zeta + 2\ell)_\ell^{(2)} + (\zeta + 3\ell)_\ell^{(2)}$$

$$= (3)_2^{(2)} + (3+2)_2^{(2)} + (3+4)_2^{(2)} + (3+6)_2^{(2)} = (3)_2^{(2)} + (5)_2^{(2)} + (7)_2^{(2)} + (9)_2^{(2)} = 116$$

$$\text{We know that, } \Delta_\ell^{-1} \zeta_\ell^{(2)} = \frac{\zeta_\ell^{(3)}}{3\ell}$$

$$\text{RHS} = \Delta_\ell^{-1} (\zeta + (m+1)\ell)_\ell^{(2)} - \Delta_\ell^{-1} (\zeta)_\ell^{(2)} = \frac{(\zeta + (m+1)\ell)_\ell^{(3)}}{3\ell} - \frac{\zeta_\ell^{(3)}}{3\ell}$$

$$= \frac{(3 + (3+1)2)_2^{(3)}}{6} - \frac{3_2^{(3)}}{6} = \frac{11.9.7}{6} - \frac{3.1.-1}{6} = 116$$

Therefore, LHS=RHS=116.

Theorem 3.4 If f be a real valued function, $\Delta_\ell^{-1} f(\zeta)$ and $\Delta_\ell^{-2} f(\zeta)$ exists, then

$$\sum_{r=0}^m (r+1)f(\zeta + (m-r)\ell) = \Delta_\ell^{-2} f(\zeta + (m+2)\ell) - \Delta_\ell^{-2} f(\zeta) - (m+2)\Delta_\ell^{-1} f(\zeta), \forall m \in \mathbb{Z}^+. \quad (33)$$

Proof: By equation (31), we have

$$\sum_{r=0}^m f(\zeta + (m-r)\ell) = \Delta_\ell^{-1} f(\zeta + (m+1)\ell) - \Delta_\ell^{-1} f(\zeta)$$

Substitute Δ_ℓ^{-1} on both sides,

$$\Delta_\ell^{-1} f(\zeta) + \Delta_\ell^{-1} f(\zeta + \ell) + \dots + \Delta_\ell^{-1} f(\zeta + m\ell) = \Delta_\ell^{-2} f(\zeta + (m+1)\ell) - \Delta_\ell^{-2} f(\zeta)$$

Put $m = 3$ in equation (31),

$$f(\zeta) + f(\zeta + \ell) + f(\zeta + 2\ell) + f(\zeta + 3\ell) = \Delta_\ell^{-1} f(\zeta + 4\ell) - \Delta_\ell^{-1} f(\zeta) \quad (34)$$

Put $m = 2$ in equation (31),

$$f(\zeta) + f(\zeta + \ell) + f(\zeta + 2\ell) = \Delta_\ell^{-1} f(\zeta + 3\ell) - \Delta_\ell^{-1} f(\zeta) \quad (35)$$

Put $m = 1$ in equation (31),

$$f(\zeta) + f(\zeta + \ell) = \Delta_\ell^{-1} f(\zeta + 2\ell) - \Delta_\ell^{-1} f(\zeta) \quad (36)$$

Put $m = 0$ in equation (31),

$$f(\zeta) = \Delta_\ell^{-1} f(\zeta + \ell) - \Delta_\ell^{-1} f(\zeta) \quad (37)$$

$$0 = \Delta_\ell^{-1} f(\zeta) - \Delta_\ell^{-1} f(\zeta) \quad (38)$$

Adding (34) to (38), we get

In general,

$$\begin{aligned} 4f(\zeta) + 3f(\zeta + \ell) + 2f(\zeta + 2\ell) + f(\zeta + 3\ell) &= \Delta_\ell^{-2} f(\zeta + 5\ell) - \Delta_\ell^{-2} f(\zeta) - (5)\Delta_\ell^{-1} f(\zeta), \\ (m+1)f(\zeta) + mf(\zeta + \ell) + (m-1)f(\zeta + 2\ell) + \dots + (1)f(\zeta + m\ell) \\ &= \Delta_\ell^{-2} f(\zeta + (m+2)\ell) - \Delta_\ell^{-2} f(\zeta) - (m+2)\Delta_\ell^{-1} f(\zeta). \end{aligned}$$

From the above result, we get the equation (33).

Example 3.5 Applying $f(\zeta) = \zeta_\ell^{(2)}$, $\zeta = 3$, $\ell = 2$ and $m = 3$ in equation (33), then we have

$$\begin{aligned} \text{LHS} &= (\zeta + 3\ell)_\ell^{(2)} + 2(\zeta + 2\ell)_\ell^{(2)} + 3(\zeta + \ell)_\ell^{(2)} + 4(\zeta)_\ell^{(2)} \\ &= (3 + 3(2))_2^{(2)} + 2(3 + 2(2))_2^{(2)} + 3(3 + 1(2))_2^{(2)} + 4(3)_2^{(2)} = 190. \end{aligned}$$

$$\text{We know that, } \Delta^{-1} \zeta_\ell^{(2)} = \frac{\zeta_\ell^{(3)}}{3\ell} \text{ and } \Delta^{-2} \zeta_\ell^{(2)} = \frac{\zeta_\ell^{(4)}}{12\ell^2}$$

$$\begin{aligned} \text{RHS} &= \Delta_\ell^{-2} (\zeta + (m+2)\ell) - \Delta_\ell^{-2} (\zeta) - (m+2)\Delta_\ell^{-1} f(\zeta) \\ &= \frac{(\zeta + (m+2)\ell)_\ell^{(4)}}{12\ell^2} - \frac{\zeta_\ell^{(4)}}{12\ell^2} - \frac{\zeta_\ell^{(2)}}{3\ell} = \frac{(3 + (3+2)2)_2^{(4)}}{48} - \frac{3_2^{(4)}}{48} - \frac{3_2^{(3)}}{6} = 190. \end{aligned}$$

Therefore, LHS=RHS=190.

Theorem 3.6 If f be an real valued function and $\Delta_\ell^{-1} f(\zeta)$, $\Delta_\ell^{-2} f(\zeta)$ and $\Delta_\ell^{-3} f(\zeta)$ exists, then

$$\begin{aligned} \sum_{r=0}^m \frac{(r+2)^{(2)}}{2} f(\zeta + (m-r)\ell) &= \Delta_\ell^{-3} f(\zeta + (m+3)\ell) - \Delta_\ell^{-3} f(\zeta) - (m+3)\Delta_\ell^{-2} f(\zeta) \\ &\quad - \frac{(m+3)^{(2)}}{2} \Delta_\ell^{-1} f(\zeta), \forall m \in \mathbb{Z}^+. \end{aligned} \quad (39)$$

Proof: By equation (33), then we have

$$\sum_{r=0}^m (r+1)f(\zeta + (m-r)\ell) = \Delta_\ell^{-2}f(\zeta + (m+2)\ell) - \Delta_\ell^{-2}f(\zeta) - (m+2)\Delta_\ell^{-1}f(\zeta)$$

Put $m=4$ in equation (33),

$$f(\zeta + 4\ell) + 2f(\zeta + 3\ell) + 3f(\zeta + 2\ell) + 4f(\zeta + \ell) + 5f(\zeta) = \Delta_\ell^{-2}f(\zeta + 6\ell) - \Delta_\ell^{-2}f(\zeta) - (6)\Delta_\ell^{-1}f(\zeta) \quad (40)$$

Put $m=3$ in equation (33),

$$f(\zeta + 3\ell) + 2f(\zeta + 2\ell) + 3f(\zeta + \ell) + 4f(\zeta) = \Delta_\ell^{-2}f(\zeta + 5\ell) - \Delta_\ell^{-2}f(\zeta) - (5)\Delta_\ell^{-1}f(\zeta) \quad (41)$$

Put $m=2$ in equation (33),

$$f(\zeta + 2\ell) + 2f(\zeta + \ell) + 3f(\zeta) = \Delta_\ell^{-2}f(\zeta + 4\ell) - \Delta_\ell^{-2}f(\zeta) - (4)\Delta_\ell^{-1}f(\zeta) \quad (42)$$

Put $m=1$ in equation (33),

$$f(\zeta + \ell) + 2f(\zeta) = \Delta_\ell^{-2}f(\zeta + 3\ell) - \Delta_\ell^{-2}f(\zeta) - (3)\Delta_\ell^{-1}f(\zeta) \quad (43)$$

Put $m=0$ in equation (33),

$$f(\zeta) = \Delta_\ell^{-2}f(\zeta + 2\ell) - \Delta_\ell^{-2}f(\zeta) - (2)\Delta_\ell^{-1}f(\zeta) \quad (44)$$

$$0 = \Delta_\ell^{-2}f(\zeta + \ell) - \Delta_\ell^{-2}f(\zeta) - (1)\Delta_\ell^{-1}f(\zeta) \quad (45)$$

$$0 = \Delta_\ell^{-2}f(\zeta) - \Delta_\ell^{-2}f(\zeta) - (0)\Delta_\ell^{-1}f(\zeta) \quad (46)$$

Adding (40) to (46), we get

In general,

$$\begin{aligned} & f(\zeta + 4\ell) + (1+2)f(\zeta + 3\ell) + (1+2+3)f(\zeta + 2\ell) + (1+2+3+4)f(\zeta + \ell) \\ & \quad + (1+2+3+4+5)f(\zeta) \\ & = \Delta_\ell^{-3}f(\zeta + (m+3)\ell) - \Delta_\ell^{-3}f(\zeta) - (m+3)\Delta_\ell^{-2}f(\zeta) - \frac{(7)^{(2)}}{2}\Delta_\ell^{-1}f(\zeta) \end{aligned}$$

From the above result, we get the equation (39).

Example 3.7 Applying $f(\zeta) = \zeta_\ell^{(2)}$, $\zeta = 3$, $\ell = 2$ and $m = 3$ in equation (39), we have

$$\text{LHS} = \sum_{r=0}^3 \frac{(r+2)^{(2)}}{2} f(\zeta + (3-r)\ell)$$

$$= \frac{(2)^{(2)}}{2} (3+6)_2^{(2)} + \frac{(3)^{(2)}}{2} (3+4)_2^{(2)} + \frac{(4)^{(2)}}{2} (3+2)_2^{(2)} + \frac{(5)^{(2)}}{2} (3)_2^{(2)} = 288.$$

We know that, $\Delta^{-1}\zeta_\ell^{(2)} = \frac{\zeta_\ell^{(3)}}{3\ell}$, $\Delta^{-2}\zeta_\ell^{(2)} = \frac{\zeta_\ell^{(4)}}{12\ell^2}$ and $\Delta^{-3}\zeta_\ell^{(2)} = \frac{\zeta_\ell^{(5)}}{60\ell^3}$

$$\begin{aligned} \text{RHS} &= \frac{(3+(3+3)(2))_2^{(5)}}{60\ell^3} - \frac{(3)_2^{(5)}}{60\ell^3} - 6 \frac{(3)_2^{(4)}}{12\ell^2} - \frac{(6)^{(2)}(3)_2^{(3)}}{2 \cdot 3\ell} \\ &= \frac{(15)_2^{(5)}}{480} - \frac{3_2^{(5)}}{480} - (6) \frac{3_2^{(4)}}{48} - (15) \frac{3_2^{(3)}}{6} = 288. \end{aligned}$$

Therefore, LHS=RHS=288.

Theorem 3.8 If f be a real valued function and $\Delta_\ell^{-1}f(\zeta)$, $\Delta_\ell^{-2}f(\zeta)$, $\Delta_\ell^{-3}f(\zeta)$ and $\Delta_\ell^{-4}f(\zeta)$ exists, then

$$\begin{aligned} \sum_{r=0}^m \frac{(r+3)^{(3)}}{3^{(2)}} f(\zeta + (m-r)\ell) &= \Delta_\ell^{-4}f(\zeta + (m+4)\ell) - \Delta_\ell^{-4}f(\zeta) - (m+4)\Delta_\ell^{-3}f(\zeta) \\ &\quad - \frac{(m+4)^{(2)}}{2} \Delta_\ell^{-2}f(\zeta) - \frac{(m+4)^{(3)}}{3^{(2)}} \Delta_\ell^{-1}f(\zeta), \forall m \in \mathbb{Z}^+. \end{aligned} \quad (47)$$

Proof: By equation (39), we have

$$\begin{aligned} \sum_{r=0}^m \frac{(r+2)^{(2)}}{2} f(\zeta + (m-r)\ell) &= \Delta_\ell^{-3}f(\zeta + (m+3)\ell) - \Delta_\ell^{-3}f(\zeta) - (m+3)\Delta_\ell^{-2}f(\zeta) \\ &\quad - \frac{(m+3)^{(2)}}{2} \Delta_\ell^{-1}f(\zeta) \end{aligned}$$

Put $m = 4$ in equation (39),

$$\begin{aligned} \sum_{r=0}^3 \frac{(r+2)^{(2)}}{2} f(\zeta + (4-r)\ell) &= \Delta_\ell^{-3}f(\zeta + (4+3)\ell) - \Delta_\ell^{-3}f(\zeta) - (4+3)\Delta_\ell^{-2}f(\zeta) \\ &\quad - \frac{(4+3)^{(2)}}{2} \Delta_\ell^{-1}f(\zeta), \\ \frac{(2)^{(2)}}{2} f(\zeta + 4\ell) + \frac{(3)^{(2)}}{2} f(\zeta + 3\ell) + \frac{(4)^{(2)}}{2} f(\zeta + 2\ell) + \frac{(5)^{(2)}}{2} f(\zeta + \ell) + \frac{(6)^{(2)}}{2} f(\zeta) \\ &= \Delta_\ell^{-3}f(\zeta + 6\ell) - \Delta_\ell^{-3}f(\zeta) - (6)\Delta_\ell^{-2}f(\zeta) - \frac{(6)^{(2)}}{2} \Delta_\ell^{-1}f(\zeta) \\ f(\zeta + 4\ell) + (3)f(\zeta + 3\ell) + (6)f(\zeta + 2\ell) + (10)f(\zeta + \ell) + (15)f(\zeta) \\ &= \Delta_\ell^{-3}f(\zeta + 7\ell) - \Delta_\ell^{-3}f(\zeta) - (7)\Delta_\ell^{-2}f(\zeta) - \frac{(7)^{(2)}}{2} \Delta_\ell^{-1}f(\zeta) \end{aligned} \quad (48)$$

Put $m = 3$ in equation (39),

$$\begin{aligned} \sum_{r=0}^3 \frac{(r+2)^{(2)}}{2} f(\zeta + (3-r)\ell) &= \Delta_\ell^{-3} f(\zeta + (3+3)\ell) - \Delta_\ell^{-3} f(\zeta) - (3+3)\Delta_\ell^{-2} f(\zeta) \\ &\quad - \frac{(3+3)^{(2)}}{2} \Delta_\ell^{-1} f(\zeta), \\ \frac{(2)^{(2)}}{2} f(\zeta + 3\ell) + \frac{(3)^{(2)}}{2} f(\zeta + 2\ell) + \frac{(4)^{(2)}}{2} f(\zeta + \ell) + \frac{(5)^{(2)}}{2} f(\zeta) \\ &= \Delta_\ell^{-3} f(\zeta + 6) - \Delta_\ell^{-3} f(\zeta) - (6)\Delta_\ell^{-2} f(\zeta) - \frac{(6)^{(2)}}{2} \Delta_\ell^{-1} f(\zeta), \\ f(\zeta + 3\ell) + (3)f(\zeta + 2\ell) + (6)f(\zeta + \ell) + (10)f(\zeta) \\ &= \Delta_\ell^{-3} f(\zeta + 6\ell) - \Delta_\ell^{-3} f(\zeta) - (6)\Delta_\ell^{-2} f(\zeta) - \frac{(6)^{(2)}}{2} \Delta_\ell^{-1} f(\zeta) \quad (49) \end{aligned}$$

Put $m = 2$ in equation (39),

$$\begin{aligned} \sum_{r=0}^2 \frac{(r+2)^{(2)}}{2} f(\zeta + (2-r)\ell) &= \Delta_\ell^{-3} f(\zeta + (2+3)\ell) - \Delta_\ell^{-3} f(\zeta) - (2+3)\Delta_\ell^{-2} f(\zeta) \\ &\quad - \frac{(2+3)^{(2)}}{2} \Delta_\ell^{-1} f(\zeta), \\ \frac{(2)^{(2)}}{2} f(\zeta + 2\ell) + \frac{(3)^{(2)}}{2} f(\zeta + \ell) + \frac{(4)^{(2)}}{2} f(\zeta) \\ &= \Delta_\ell^{-3} f(\zeta + 5\ell) - \Delta_\ell^{-3} f(\zeta) - (5)\Delta_\ell^{-2} f(\zeta) - \frac{(5)^{(2)}}{2} \Delta_\ell^{-1} f(\zeta), \\ f(\zeta + 2\ell) + (3)f(\zeta + \ell) + (6)f(\zeta) \\ &= \Delta_\ell^{-3} f(\zeta + 5\ell) - \Delta_\ell^{-3} f(\zeta) - (5)\Delta_\ell^{-2} f(\zeta) - \frac{(5)^{(2)}}{2} \Delta_\ell^{-1} f(\zeta) \quad (50) \end{aligned}$$

Put $m = 1$ in equation (39),

$$\begin{aligned} \sum_{r=0}^1 \frac{(r+2)^{(2)}}{2} f(\zeta + (1-r)\ell) &= \Delta_\ell^{-3} f(\zeta + (1+3)\ell) - \Delta_\ell^{-3} f(\zeta) - (1+3)\Delta_\ell^{-2} f(\zeta) \\ &\quad - \frac{(1+3)^{(2)}}{2} \Delta_\ell^{-1} f(\zeta), \\ \frac{(2)^{(2)}}{2} f(\zeta + \ell) + \frac{(3)^{(2)}}{2} f(\zeta) &= \Delta_\ell^{-3} f(\zeta + 4\ell) - \Delta_\ell^{-3} f(\zeta) - (4)\Delta_\ell^{-2} f(\zeta) - \frac{(4)^{(2)}}{2} \Delta_\ell^{-1} f(\zeta), \\ f(\zeta + \ell) + (3)f(\zeta) &= \Delta_\ell^{-3} f(\zeta + 4\ell) - \Delta_\ell^{-3} f(\zeta) - (4)\Delta_\ell^{-2} f(\zeta) - \frac{(4)^{(2)}}{2} \Delta_\ell^{-1} f(\zeta) \quad (51) \end{aligned}$$

Put $m = 0$ in equation (39),

$$\begin{aligned} \frac{(2)^{(2)}}{2} f(\zeta) &= \Delta_\ell^{-3} f(\zeta + 3\ell) - \Delta_\ell^{-3} f(\zeta) - (3)\Delta_\ell^{-2} f(\zeta) - \frac{(3)^{(2)}}{2} \Delta_\ell^{-1} f(\zeta) \\ f(\zeta) &= \Delta_\ell^{-3} f(\zeta + 3\ell) - \Delta_\ell^{-3} f(\zeta) - (3)\Delta_\ell^{-2} f(\zeta) - \frac{(3)^{(2)}}{2} \Delta_\ell^{-1} f(\zeta) \end{aligned} \quad (52)$$

$$0 = \Delta_\ell^{-3} f(\zeta + 2\ell) - \Delta_\ell^{-3} f(\zeta) - (2)\Delta_\ell^{-2} f(\zeta) - \frac{(2)^{(2)}}{2} \Delta_\ell^{-1} f(\zeta) \quad (53)$$

$$0 = \Delta_\ell^{-3} f(\zeta + \ell) - \Delta_\ell^{-3} f(\zeta) - (1)\Delta_\ell^{-2} f(\zeta) - \frac{(1)^{(2)}}{2} \Delta_\ell^{-1} f(\zeta) \quad (54)$$

$$0 = \Delta_\ell^{-3} f(\zeta) - \Delta_\ell^{-3} f(\zeta) \quad (55)$$

Adding (48) to (55), we get

$$\begin{aligned} f(\zeta + 4\ell) + (1+3)f(\zeta + 3\ell) + (1+3+6)f(\zeta + 2\ell) + (1+3+6+10)f(\zeta + \ell) \\ + (1+3+6+10+15)f(\zeta) \\ = \Delta_\ell^{-4} f(\zeta + (m+4)\ell) - \Delta_\ell^{-4} f(\zeta) - (m+4)\Delta_\ell^{-3} f(\zeta) - \frac{(m+4)^{(2)}}{2} \Delta_\ell^{-2} f(\zeta) \\ - (1+3+6+10+15+21)\Delta_\ell^{-1} f(\zeta) \end{aligned}$$

In general,

$$\begin{aligned} f(\zeta + 4\ell) + (4)f(\zeta + 3\ell) + (10)f(\zeta + 2\ell) + (20)f(\zeta + \ell) + (35)f(\zeta) \\ = \Delta_\ell^{-4} f(\zeta + (m+4)\ell) - \Delta_\ell^{-4} f(\zeta) - (m+4)\Delta_\ell^{-3} f(\zeta) \\ - \frac{(m+4)^{(2)}}{2} \Delta_\ell^{-2} f(\zeta) - (56)\Delta_\ell^{-1} f(\zeta) \end{aligned}$$

From the above result, we get the equation (47).

Example 3.9 Applying $f(\zeta) = \zeta_\ell^{(2)}$, $\zeta = 3$, $\ell = 2$ and $m=3$ in equation (47), then we have

$$\begin{aligned} \text{LHS} &= \sum_{r=0}^3 \frac{(r+3)^{(3)}}{3^{(2)}} f(\zeta + (3-r)\ell) \\ &= \frac{(3)^{(3)}}{3^{(2)}} (3+6)_\ell^{(2)} + \frac{(4)^{(3)}}{3^{(2)}} (3+4)_\ell^{(2)} + \frac{(5)^{(3)}}{3^{(2)}} (3+2)_\ell^{(2)} + \frac{(6)^{(3)}}{3^{(2)}} (3)_\ell^{(2)} = 413. \end{aligned}$$

We know that, $\Delta^{-1}\zeta_\ell^{(2)} = \frac{\zeta_\ell^{(3)}}{3\ell}$, $\Delta^{-2}\zeta_\ell^{(2)} = \frac{\zeta_\ell^{(4)}}{12\ell^2}$, $\Delta^{-3}\zeta_\ell^{(2)} = \frac{\zeta_\ell^{(5)}}{60\ell^3}$ and $\Delta^{-4}\zeta_\ell^{(2)} = \frac{\zeta_\ell^{(6)}}{360\ell^4}$

$$\begin{aligned} \text{RHS} &= \frac{(3+(3+4)\ell)_\ell^{(6)}}{360\ell^4} - \frac{(3)_\ell^{(6)}}{360\ell^3} - 7\frac{(3)_\ell^{(5)}}{60\ell^3} - \frac{(7)^{(2)}(3)_\ell^{(4)}}{2 \cdot 12\ell^2} - \frac{7^{(3)} \cdot 3_\ell^{(3)}}{3^{(2)} \cdot 3\ell} \\ &= \frac{17_2^{(6)}}{5760} - \frac{3_2^{(6)}}{5760} - 7\frac{(3)_2^{(5)}}{480} - \frac{(7)^{(2)}(3)_2^{(4)}}{2 \cdot 48} - \frac{7^{(3)} \cdot 3_2^{(3)}}{3^{(2)} \cdot 6} = 413. \end{aligned}$$

Therefore, $LHS=RHS=413$.

Theorem 3.10 If f be a real valued function and $\Delta_\ell^{-r} f(\zeta)$ where $r = 1, 2, \dots, n$ exists, then

$$\sum_{r=0}^m \frac{(r+n-1)^{(n-1)}}{(n-1)!} f(\zeta + (m-r)\ell) = \Delta_\ell^{-n} f(\zeta + (m+n)\ell) - \Delta_\ell^{-n} f(\zeta) - \frac{(m+n)^{(1)}}{1!} \Delta_\ell^{-(n-1)} f(\zeta) - \frac{(m+n)^{(2)}}{2!} \Delta_\ell^{-(n-2)} f(\zeta) - \frac{(m+n)^{(n-1)}}{(n-1)!} \Delta_\ell^{-1} f(\zeta), \forall m \in \mathbb{Z}^+. \quad (56)$$

Proof: The proof follows from, put $n = 1$ in equation (56),

$$\sum_{r=0}^m f(\zeta + (m-r)\ell) = \Delta_\ell^{-1} f(\zeta + (m+1)\ell) - \Delta_\ell^{-1} f(\zeta)$$

Put $n = 2$ in equation (56),

$$\sum_{r=0}^m (r+1) f(\zeta + (m-r)\ell) = \Delta_\ell^{-2} f(\zeta + (m+2)\ell) - \Delta_\ell^{-2} f(\zeta) - (m+2) \Delta_\ell^{-1} f(\zeta)$$

Put $n = 3$ in equation (56),

$$\sum_{r=0}^m \frac{(r+2)^{(2)}}{2} f(\zeta + (m-r)\ell) = \Delta_\ell^{-3} f(\zeta + (m+3)\ell) - \Delta_\ell^{-3} f(\zeta) - (m+3) \Delta_\ell^{-2} f(\zeta) - \frac{(m+3)^{(2)}}{2!} \Delta_\ell^{-1} f(\zeta)$$

Put $n = 4$ in equation (56),

$$\sum_{r=0}^m \frac{(r+3)^{(3)}}{3!} f(\zeta + (m-r)\ell) = \Delta_\ell^{-4} f(\zeta + (m+4)\ell) - \Delta_\ell^{-4} f(\zeta) - (m+4) \Delta_\ell^{-3} f(\zeta) - \frac{(m+4)^{(2)}}{2!} \Delta_\ell^{-2} f(\zeta) - \frac{(m+4)^{(3)}}{3!} \Delta_\ell^{-1} f(\zeta).$$

Example 3.11 Put $n = 5$ by equation (56)

$$\sum_{r=0}^m \frac{(r+4)^{(4)}}{4!} f(\zeta + (m-r)\ell) = \Delta_\ell^{-5} f(\zeta + (m+5)\ell) - \Delta_\ell^{-5} f(\zeta) - \frac{(m+5)^{(1)}}{1!} \Delta_\ell^{-4} f(\zeta) - \frac{(m+5)^{(2)}}{2!} \Delta_\ell^{-3} f(\zeta) - \frac{(m+5)^{(3)}}{3!} \Delta_\ell^{-2} f(\zeta) - \frac{(m+5)^{(4)}}{4!} \Delta_\ell^{-1} f(\zeta)$$

Applying $f(\zeta) = \zeta_\ell^{(2)}$, $\zeta = 3$, $\ell = 2$ and $m = 3$ in the above equation, we get

$$\begin{aligned} \text{LHS} &= \sum_{r=0}^3 \frac{(r+3)^{(3)}}{3!} f(\zeta + (3-r)\ell) \\ &= \frac{(4)^{(4)}}{4!} (3+6)_\ell^{(2)} + \frac{(5)^{(4)}}{4!} (3+4)_\ell^{(2)} + \frac{(6)^{(4)}}{4!} (3+2)_\ell^{(2)} + \frac{(7)^{(4)}}{4!} (3)_\ell^{(2)} = 568. \end{aligned}$$

We know that, $\Delta^{-1}\zeta_\ell^{(2)} = \frac{\zeta_\ell^{(3)}}{3\ell}$, $\Delta^{-2}\zeta_\ell^{(2)} = \frac{\zeta_\ell^{(4)}}{12\ell^2}$, $\Delta^{-3}\zeta_\ell^{(2)} = \frac{\zeta_\ell^{(5)}}{60\ell^3}$, $\Delta^{-4}\zeta_\ell^{(2)} = \frac{\zeta_\ell^{(6)}}{360\ell^4}$

and $\Delta^{-5}\zeta_\ell^{(2)} = \frac{\zeta_\ell^{(7)}}{25200\ell^5}$

$$\begin{aligned} \text{RHS} &= \frac{(3+(3+5)\ell)_\ell^{(7)}}{2520\ell^5} - \frac{(3)_\ell^{(7)}}{2520\ell^4} - (8) \frac{(3)_\ell^{(6)}}{360\ell^4} - \frac{(8)^{(2)}(3)_\ell^{(5)}}{2! \cdot 60\ell^3} - \frac{8^{(3)}3_\ell^{(4)}}{3! \cdot 12\ell^2} - \frac{8^{(4)}3_\ell^{(3)}}{4! \cdot 3\ell} \\ &= \frac{19_2^{(7)}}{80640} - \frac{3_2^{(7)}}{80640} - (8) \frac{(3)_2^{(6)}}{5760} - \frac{(8)^{(2)}(3)_2^{(5)}}{2! \cdot 480} - \frac{8^{(3)}3_2^{(4)}}{3! \cdot 48} - \frac{8^{(4)}3_2^{(3)}}{4 \cdot 6} = 568. \end{aligned}$$

Therefore, LHS=RHS=568.

Thus we have obtained Delta- ℓ operator its inverse and some basic results using these operator.

4 Nabla Operator

Definition 4.1 Let f be a real valued function and $\zeta \neq 0$. The Nabla operator on f is defined as

$$\nabla f(\zeta) = f(\zeta) - f(\zeta - 1). \quad (57)$$

The inverse of nabla operator on f is defined by, if there exists $g(\zeta)$ such that

$$\nabla g(\zeta) = f(\zeta) \Leftrightarrow g(\zeta) + c = \nabla^{-1}f(\zeta), \quad (58)$$

where c is an arbitrary constant.

Theorem 4.2 If f be a real valued function and $\nabla^{-1}f(\zeta)$ exists, then

$$\sum_{r=0}^m f(\zeta + m - r) = \nabla^{-1}f(\zeta + m) - \nabla^{-1}f(\zeta - 1), \forall m \in \mathbb{Z}^+. \quad (59)$$

Proof: From equation (57), $f(\zeta) = f(\zeta) - f(\zeta - 1)$

Consider $\nabla^{-1}f(\zeta) = g(\zeta)$, then $f(\zeta) = \nabla g(\zeta)$

Since $f(\zeta) = g(\zeta) - g(\zeta - 1)$,

$$f(\zeta) + g(\zeta - 1) = g(\zeta) \quad (60)$$

Replace ζ by $\zeta + 1$ in equation (60),

$$f(\zeta + 1) + g(\zeta) = g(\zeta + 1)$$

Substitute equation (60),

$$f(\zeta + 1) + f(\zeta) + g(\zeta - 1) = g(\zeta + 1).$$

Again replace ζ by $\zeta + 1$,

$$f(\zeta + 2) + f(\zeta + 1) + g(\zeta) = g(\zeta + 2)$$

Substitute equation (60),

$$f(\zeta + 2) + f(\zeta + 1) + f(\zeta) + g(\zeta - 1) = g(\zeta + 2).$$

⋮

By induction;

$$f(\zeta + m) + f(\zeta + m - 1) + \cdots + f(\zeta) + g(\zeta - 1) = g(\zeta + m)$$

$$f(\zeta) + f(\zeta + 1) + \cdots + f(\zeta + m) = \nabla^{-1}f(\zeta + m) - \nabla^{-1}f(\zeta - 1)$$

From the above result, we get the equation (59).

Example 4.3 Applying $f(\zeta) = \zeta^{(2)}$, $\zeta = 2$ and $m = 3$ in equation (59), then we have

$$\begin{aligned} \text{LHS} &= \sum_{r=0}^m f(\zeta + m - r) = \zeta^{(2)} + (\zeta + 1)^{(2)} + (\zeta + 2)^{(2)} + (\zeta + 3)^{(2)} = 2^{(2)} + 3^{(2)} + 4^{(2)} + 5^{(2)} \\ &= 2.1 + 3.2 + 4.3 + 5.4 = 2 + 6 + 12 + 20 = 40. \end{aligned}$$

We know that $\nabla^{-1}\zeta^{(2)} = \frac{(\zeta + 1)^{(3)}}{3}$

$$\begin{aligned} \text{RHS} &= \nabla^{-1}f(\zeta + m) - \nabla^{-1}f(\zeta - 1) = \frac{(\zeta + 3 + 1)^{(3)}}{3} - \frac{(\zeta + 1 - 1)^{(3)}}{3} \\ &= \frac{(6)^{(3)}}{3} - \frac{(2)^{(3)}}{3} = \frac{6.5.4}{3} - \frac{2.1.0}{3} = 40. \end{aligned}$$

Therefore, LHS=RHS=40.

Theorem 4.4 If f be a real valued function, $\nabla^{-1}f(\zeta)$ and $\nabla^{-2}f(\zeta)$ exists, then

$$\sum_{r=0}^m (r + 1)f(\zeta + m - r) = \nabla^{-2}f(\zeta + m) - \nabla^{-2}f(\zeta - 1) - (m + 1)\nabla^{-1}f(\zeta - 1), \forall m \in \mathbb{Z}^+. \tag{61}$$

Proof: By equation (59), we have

$$\sum_{r=0}^m f(\zeta + m - r) = \nabla^{-1}f(\zeta + m) - \nabla^{-1}f(\zeta - 1)$$

Applying ∇^{-1} on both sides,

$$\nabla^{-1}f(\zeta) + \nabla^{-1}f(\zeta + 1) + \cdots + \nabla^{-1}f(\zeta + m) = \nabla^{-2}f(\zeta + m) - \nabla^{-2}f(\zeta - 1)$$

Put $m = 4$ in equation (59),

$$f(\zeta) + f(\zeta + 1) + f(\zeta + 2) + f(\zeta + 3) + f(\zeta + 4) = \nabla^{-1}f(\zeta + 4) - \nabla^{-1}f(\zeta - 1) \quad (62)$$

Put $m = 3$ in equation (59),

$$f(\zeta) + f(\zeta + 1) + f(\zeta + 2) + f(\zeta + 3) = \nabla^{-1}f(\zeta + 3) - \nabla^{-1}f(\zeta - 1) \quad (63)$$

Put $m = 2$ in equation (59),

$$f(\zeta) + f(\zeta + 1) + f(\zeta + 2) = \nabla^{-1}f(\zeta + 2) - \nabla^{-1}f(\zeta - 1) \quad (64)$$

Put $m = 1$ in equation (59),

$$f(\zeta) + f(\zeta + 1) = \nabla^{-1}f(\zeta + 1) - \nabla^{-1}f(\zeta - 1) \quad (65)$$

Put $m = 0$ in equation (59),

$$f(\zeta) = \nabla^{-1}f(\zeta) - \nabla^{-1}f(\zeta - 1) \quad (66)$$

Adding (62) to (66), we get

$$\begin{aligned} 5f(\zeta) + 4f(\zeta + 1) + 3f(\zeta + 2) + 2f(\zeta + 3) + f(\zeta + 4) \\ = \nabla^{-2}f(\zeta + 4) - \nabla^{-2}f(\zeta - 1) - (5)\nabla^{-1}f(\zeta - 1) \end{aligned}$$

In general,

$$\begin{aligned} (m + 1)f(\zeta) + mf(\zeta + 1) + (m - 1)f(\zeta + 2) + \dots + (1)f(\zeta + m) \\ = \nabla^{-2}f(\zeta + m) - \nabla^{-2}f(\zeta - 1) - (m + 1)\nabla^{-1}f(\zeta - 1) \end{aligned}$$

From the above result, we get the equation (61).

Example 4.5 Applying $f(\zeta) = \zeta^{(2)}$, $\zeta = 2$ and $m = 3$ in equation (61), then we have

$$\begin{aligned} \text{LHS} &= \sum_{r=0}^m (r + 1)f(\zeta + m - r) = (\zeta + 3)^{(2)} + 2(\zeta + 2)^{(2)} + 3(\zeta + 1)^{(2)} + 4\zeta^{(2)} \\ &= 5^{(2)} + (2)4^{(2)} + (3)3^{(2)} + (4)2^{(2)} = 5.4 + 2(4.3) + 3(3.2) + 4(2.1) = 70. \end{aligned}$$

$$\text{We know that, } \nabla^{-1}\zeta^{(2)} = \frac{(\zeta + 1)^{(3)}}{3} \text{ and } \nabla^{-2}\zeta^{(2)} = \frac{(\zeta + 2)^{(4)}}{12}.$$

$$\begin{aligned} \text{RHS} &= \nabla^{-2}f(\zeta + m) - \nabla^{-2}f(\zeta - 1) - (m + 1)\nabla^{-1}f(\zeta - 1) \\ &= \frac{(\zeta + 3 + 2)^{(4)}}{12} - \frac{(\zeta + 1 - 2)^{(4)}}{12} - (4)\frac{(\zeta + 1 - 1)^{(3)}}{3} = \frac{(7)^{(4)}}{12} - \frac{(2)^{(4)}}{12} - (4)\frac{(2)^{(3)}}{3} \end{aligned}$$

$$= 70.$$

Therefore, LHS=RHS=70.

Theorem 4.6 If f be a real valued function and $\nabla^{-1}f(\zeta)$, $\nabla^{-2}f(\zeta)$ and $\nabla^{-3}f(\zeta)$ exists, then

$$\sum_{r=0}^m \frac{(r+2)^{(2)}}{2!} f(\zeta+m-r) = \nabla^{-3}f(\zeta+m) - \nabla^{-3}f(\zeta-1) - (m+1)\nabla^{-2}f(\zeta-1) - \frac{(m+2)^{(2)}}{2} \nabla^{-1}f(\zeta-1), \forall m \in \mathbb{Z}^+. \quad (67)$$

Proof: By equation (61), then we have

$$\sum_{r=0}^m (r+1)f(\zeta+m-r) = \nabla^{-2}f(\zeta+m) - \nabla^{-2}f(\zeta-1) - (m+1)\nabla^{-1}f(\zeta-1)$$

Put $m = 4$ in equation (61),

$$f(\zeta+4) + 2f(\zeta+3) + 3f(\zeta+2) + 4f(\zeta+1) + 5f(\zeta) = \nabla^{-2}f(\zeta+4) - \nabla^{-2}f(\zeta-1) - (5)\nabla^{-1}f(\zeta-1) \quad (68)$$

Put $m = 3$ in equation (61),

$$f(\zeta+3) + 2f(\zeta+2) + 3f(\zeta+1) + 4f(\zeta) = \nabla^{-2}f(\zeta+3) - \nabla^{-2}f(\zeta-1) - (4)\nabla^{-1}f(\zeta-1) \quad (69)$$

Put $m = 2$ in equation (61),

$$f(\zeta+2) + 2f(\zeta+1) + 3f(\zeta) = \nabla^{-2}f(\zeta+2) - \nabla^{-2}f(\zeta-1) - (3)\nabla^{-1}f(\zeta-1) \quad (70)$$

Put $m = 1$ in equation (61),

$$f(\zeta+1) + 2f(\zeta) = \nabla^{-2}f(\zeta+1) - \nabla^{-2}f(\zeta-1) - (2)\nabla^{-1}f(\zeta-1) \quad (71)$$

Put $m = 0$ in equation (61),

$$f(\zeta) = \nabla^{-2}f(\zeta) - \nabla^{-2}f(\zeta-1) - (1)\nabla^{-1}f(\zeta-1) \quad (72)$$

Addind (68) to (72), we get

In general,

$$f(\zeta+4) + (1+2)f(\zeta+3) + (1+2+3)f(\zeta+2) + (1+2+3+4)f(\zeta+1) + (1+2+3+4+5)f(\zeta)$$

$$= \nabla^{-3}f(\zeta + m) - \nabla^{-3}f(\zeta - 1) - (m + 1)\nabla^{-2}f(\zeta - 1) - \frac{(6)^{(2)}}{2}\nabla^{-1}f(\zeta - 1)$$

From the above result, we get the equation (67).

Example 4.7 Applying $f(\zeta) = \zeta^{(2)}$, $\zeta = 2$ and $m = 3$ in equation (67), then we have

$$\begin{aligned} \text{LHS} &= \sum_{r=0}^m \frac{(r+2)^{(2)}}{2} f(\zeta + m - r) = (\zeta + 3)^{(2)} + 3(\zeta + 2)^{(2)} + 6(\zeta + 1)^{(2)} + 10\zeta^{(2)} \\ &= 5^{(2)} + (3)4^{(2)} + (6)3^{(2)} + (10)2^{(2)} = 5.4 + 3(4.3) + 6(3.2) + 10(2.1) = 112. \end{aligned}$$

$$\text{We know that, } \nabla^{-1}\zeta^{(2)} = \frac{(\zeta + 1)^{(3)}}{3}, \nabla^{-2}\zeta^{(2)} = \frac{(\zeta + 2)^{(4)}}{12} \text{ and } \nabla^{-3}\zeta^{(2)} = \frac{(\zeta + 3)^{(5)}}{60}$$

$$\begin{aligned} \text{RHS} &= \nabla^{-3}f(\zeta + m) - \nabla^{-3}f(\zeta - 1) - (m + 1)\nabla^{-2}f(\zeta - 1) - \frac{(m+2)^{(2)}}{2}\nabla^{-1}f(\zeta - 1) \\ &= \frac{(\zeta + 3 + 3)^{(5)}}{60} - \frac{(\zeta + 3 - 1)^{(5)}}{60} - (4)\frac{(\zeta + 2 - 1)^{(4)}}{12} - (10)\frac{(\zeta + 1 - 1)^{(3)}}{3} = 112. \end{aligned}$$

Therefore, LHS=RHS=112.

Theorem 4.8 If f be a real valued function and $\nabla^{-1}f(\zeta)$, $\nabla^{-2}f(\zeta)$, $\nabla^{-3}f(\zeta)$ and $\nabla^{-4}f(\zeta)$ exists, then

$$\begin{aligned} \sum_{r=0}^m \frac{(r+3)^{(3)}}{3!} f(\zeta + m - r) &= \nabla^{-4}f(\zeta + m) - \nabla^{-4}f(\zeta - 1) - (m + 1)\nabla^{-3}f(\zeta - 1) \\ &\quad - \frac{(m+2)^{(2)}}{2!}\nabla^{-2}f(\zeta - 1) \\ &\quad - \frac{(m+3)^{(3)}}{3!}\nabla^{-1}f(\zeta - 1), \forall m \in \mathbb{Z}^+. \end{aligned} \quad (73)$$

Proof: By equation (67), we have

$$\begin{aligned} \sum_{r=0}^m \frac{(r+2)^{(2)}}{2} f(\zeta + m - r) &= \nabla^{-3}f(\zeta + m) - \nabla^{-3}f(\zeta - 1) - (m + 1)\nabla^{-2}f(\zeta - 1) \\ &\quad - \frac{(m+2)^{(2)}}{2}\nabla^{-1}f(\zeta - 1) \end{aligned}$$

Put $m = 4$ in equation (67),

$$\begin{aligned} \sum_{r=0}^4 \frac{(r+2)^{(2)}}{2} f(\zeta + 4 - r) &= \nabla^{-3}f(\zeta + 4) - \nabla^{-3}f(\zeta - 1) - (4 + 1)\nabla^{-2}f(\zeta - 1) \\ &\quad - \frac{(4+2)^{(2)}}{2}\nabla^{-1}f(\zeta - 1), \\ \frac{(2)^{(2)}}{2}f(\zeta + 4) + \frac{(3)^{(2)}}{2}f(\zeta + 3) + \frac{(4)^{(2)}}{2}f(\zeta + 2) + \frac{(5)^{(2)}}{2}f(\zeta + 1) + \frac{(6)^{(2)}}{2}f(\zeta) \end{aligned}$$

$$\begin{aligned}
 &= \nabla^{-3}f(\zeta + 4) - \nabla^{-3}f(\zeta - 1) - (5)\nabla^{-2}f(\zeta - 1) - \frac{(6)^{(2)}}{2}\nabla^{-1}f(\zeta - 1) \\
 &f(\zeta + 4) + (3)f(\zeta + 3) + (6)f(\zeta + 2) + (10)f(\zeta + 1) + (15)f(\zeta) \\
 &= \nabla^{-3}f(\zeta + 4) - \nabla^{-3}f(\zeta - 1) - (5)\nabla^{-2}f(\zeta - 1) - (15)\nabla^{-1}f(\zeta - 1), \\
 &\hspace{20em} (74)
 \end{aligned}$$

Put $m = 3$ in equation (67),

$$\begin{aligned}
 \sum_{r=0}^3 \frac{(r+2)^{(2)}}{2} f(\zeta + 3 - r) &= \nabla^{-3}f(\zeta + 3) - \nabla^{-3}f(\zeta - 1) - (3+1)\nabla^{-2}f(\zeta - 1) \\
 &\quad - \frac{(3+2)^{(2)}}{2}\nabla^{-1}f(\zeta - 1), \\
 \frac{(2)^{(2)}}{2}f(\zeta + 3) + \frac{(3)^{(2)}}{2}f(\zeta + 2) + \frac{(4)^{(2)}}{2}f(\zeta + 1) + \frac{(5)^{(2)}}{2}f(\zeta) \\
 &= \nabla^{-3}f(\zeta + 3) - \nabla^{-3}f(\zeta - 1) - (4)\nabla^{-2}f(\zeta - 1) - \frac{(5)^{(2)}}{2}\nabla^{-1}f(\zeta - 1), \\
 &f(\zeta + 3) + (3)f(\zeta + 2) + (6)f(\zeta + 1) + (10)f(\zeta) \\
 &= \nabla^{-3}f(\zeta + 3) - \nabla^{-3}f(\zeta - 1) - (4)\nabla^{-2}f(\zeta - 1) - (10)\nabla^{-1}f(\zeta - 1) \\
 &\hspace{20em} (75)
 \end{aligned}$$

Put $m = 2$ in equation (67),

$$\begin{aligned}
 \sum_{r=0}^2 \frac{(r+2)^{(2)}}{2} f(\zeta + 2 - r) &= \nabla^{-3}f(\zeta + 2) - \nabla^{-3}f(\zeta - 1) - (2+1)\nabla^{-2}f(\zeta - 1) \\
 &\quad - \frac{(2+2)^{(2)}}{2}\nabla^{-1}f(\zeta - 1), \\
 \frac{(2)^{(2)}}{2}f(\zeta + 2) + \frac{(3)^{(2)}}{2}f(\zeta + 1) + \frac{(4)^{(2)}}{2}f(\zeta) \\
 &= \nabla^{-3}f(\zeta + 2) - \nabla^{-3}f(\zeta - 1) - (3)\nabla^{-2}f(\zeta - 1) - \frac{(4)^{(2)}}{2}\nabla^{-1}f(\zeta - 1)
 \end{aligned}$$

$$f(\zeta + 2) + (3)f(\zeta + 1) + (6)f(\zeta) = \nabla^{-3}f(\zeta + 2) - \nabla^{-3}f(\zeta - 1) - (3)\nabla^{-2}f(\zeta - 1) - (6)\nabla^{-1}f(\zeta - 1) \quad (76)$$

Put $m = 1$ in equation (67),

$$\begin{aligned}
 \sum_{r=0}^1 \frac{(r+2)^{(2)}}{2} f(\zeta + 1r) &= \nabla^{-3}f(\zeta + 1) - \nabla^{-3}f(\zeta - 1) - (1+1)\nabla^{-2}f(\zeta - 1) \\
 &\quad - \frac{(1+2)^{(2)}}{2}\nabla^{-1}f(\zeta - 1) \\
 \frac{(2)^{(2)}}{2}f(\zeta + 1) + \frac{(3)^{(2)}}{2}f(\zeta) &= \nabla^{-3}f(\zeta + 1) - \nabla^{-3}f(\zeta - 1) - (2)\nabla^{-2}f(\zeta - 1)
 \end{aligned}$$

$$-\frac{(3)^{(2)}}{2} \nabla^{-1} f(\zeta - 1)$$

$$f(\zeta + 1) + (3)f(\zeta) = \nabla^{-3} f(\zeta + 1) - \nabla^{-3} f(\zeta - 1) - (2)\nabla^{-2} f(\zeta - 1) - (3)\nabla^{-1} f(\zeta - 1) \quad (77)$$

Put $m = 0$ in equation (67),

$$\begin{aligned} \frac{(2)^{(2)}}{2} f(\zeta) &= \nabla^{-3} f(\zeta) - \nabla^{-3} f(\zeta - 1) - (1)\nabla^{-2} f(\zeta - 1) - \frac{(2)^{(2)}}{2} \nabla^{-1} f(\zeta - 1) \\ f(\zeta) &= \nabla^{-3} f(\zeta) - \nabla^{-3} f(\zeta - 1) - \nabla^{-2} f(\zeta - 1) - \nabla^{-1} f(\zeta - 1) \end{aligned} \quad (78)$$

Addind (74) to (78), we get

$$\begin{aligned} f(\zeta + 4) + (1 + 3)f(\zeta + 3) + (1 + 3 + 6)f(\zeta + 2) + (1 + 3 + 6 + 10)f(\zeta + 1) \\ - (+ (1 + 3 + 6 + 10 + 15)f(\zeta) \\ = \nabla^{-4} f(\zeta + m) - \nabla^{-4} f(\zeta - 1) - (m + 1)\nabla^{-3} f(\zeta - 1) \\ - (1 + 2 + 3 + 4 + 5)\nabla^{-2} f(\zeta - 1) \\ - (1 + 3 + 6 + 10 + 15)\nabla^{-1} f(\zeta - 1) \end{aligned}$$

In general,

$$\begin{aligned} f(\zeta + 4) + (4)f(\zeta + 3) + (10)f(\zeta + 2) + (20)f(\zeta + 1) + (35)f(\zeta) \\ = \nabla^{-4} f(\zeta + m) - \nabla^{-4} f(\zeta - 1) - (m + 1)\nabla^{-3} f(\zeta - 1) \\ - (15)\nabla^{-2} f(\zeta - 1) - (35)\nabla^{-1} f(\zeta - 1) \end{aligned}$$

From the above result, we get the equation (73).

Example 4.9 Applying $f(\zeta) = \zeta^{(2)}$, $\zeta = 2$ and $m = 3$ in equation (73), then we have

$$\begin{aligned} \text{LHS} &= \sum_{r=0}^m \frac{(r+3)^{(3^1)}}{2} f(\zeta + m - r) = (\zeta + 3)^{(2)} + 4(\zeta + 2)^{(2)} + 10(\zeta + 1)^{(2)} + 20\zeta^{(2)} \\ &= 5^{(2)} + (4)4^{(2)} + (10)3^{(2)} + (20)2^{(2)} = 5.4 + 4(4.3) + 10(3.2) + 20(2.1) = 168. \end{aligned}$$

We know that, $\nabla^{-1} \zeta^{(2)} = \frac{(\zeta + 1)^{(3)}}{3}$, $\nabla^{-2} \zeta^{(2)} = \frac{(\zeta + 2)^{(4)}}{12}$, $\nabla^{-3} \zeta^{(2)} = \frac{(\zeta + 3)^{(5)}}{60}$ and

$$\nabla^{-4} \zeta^{(2)} = \frac{(\zeta + 4)^{(6)}}{360}$$

$$\begin{aligned} \text{RHS} &= \frac{(\zeta + 4 + 3)^{(6)}}{360} - \frac{(\zeta + 4 - 1)^{(6)}}{360} - (4) \frac{(\zeta + 3 - 1)^{(5)}}{60} - (10) \frac{(\zeta + 2 - 1)^{(4)}}{12} - \\ &(20) \frac{(\zeta + 1 - 1)^{(3)}}{3} \\ &= \frac{(9)^{(6)}}{360} - \frac{(5)^{(6)}}{360} - (4) \frac{(4)^{(5)}}{60} - (10) \frac{(3)^{(4)}}{12} - (20) \frac{(2)^{(3)}}{3} = 168. \end{aligned}$$

Therefore, LHS=RHS=168.

Theorem 4.10 If f be an real valued function and $\nabla^{-r}f(\zeta)$ where $r = 1, 2, \dots, n$ exists, then

$$\begin{aligned} \sum_{r=0}^m \frac{(r+n-1)^{(n-1)}}{(n-1)!} f(\zeta+m-r) &= \nabla^{-n}f(\zeta+m) - \nabla^{-n}f(\zeta-1) \\ &\quad - \frac{(m+1)^{(1)}}{1!} \nabla^{-(n-1)}f(\zeta-1) \\ &\quad - \frac{(m+2)^{(2)}}{2!} \nabla^{-(n-2)}f(\zeta-1) \\ &\quad - \frac{(m+n-1)^{(n-1)}}{(n-1)!} \Delta^{-1}f(\zeta-1), \forall m \in \mathbb{Z}^+. \end{aligned} \quad (79)$$

Proof: The proof follows from, put $n = 1$ in equation (79),

$$\sum_{r=0}^m f(\zeta+m-r) = \nabla^{-1}f(\zeta+m) - \nabla^{-1}f(\zeta-1)$$

Put $n = 2$ in equation (79)

$$\sum_{r=0}^m (r+1)f(\zeta+m-r) = \nabla^{-2}f(\zeta+m) - \nabla^{-2}f(\zeta-1) - (m+1)\nabla^{-1}f(\zeta-1)$$

Put $n = 3$ in equation (79),

$$\begin{aligned} \sum_{r=0}^m \frac{(r+2)^{(2)}}{2!} f(\zeta+m-r) &= \nabla^{-3}f(\zeta+m) - \nabla^{-3}f(\zeta-1) - (m+1)\nabla^{-2}f(\zeta-1) \\ &\quad - \frac{(m+2)^{(2)}}{2} \nabla^{-1}f(\zeta-1) \end{aligned}$$

Put $n = 4$ in equation (79),

$$\begin{aligned} \sum_{r=0}^m \frac{(r+3)^{(3)}}{3!} f(\zeta+m-r) &= \nabla^{-4}f(\zeta+m) - \nabla^{-4}f(\zeta-1) - (m+1)\nabla^{-3}f(\zeta-1) \\ &\quad - \frac{(m+2)^{(2)}}{2!} \nabla^{-2}f(\zeta-1) - \frac{(m+3)^{(3)}}{3!} \nabla^{-1}f(\zeta-1) \end{aligned}$$

Example 4.11 Put $n = 5$ by equation (79)

$$\begin{aligned} \sum_{r=0}^m \frac{(r+4)^{(4)}}{4!} f(\zeta+m-r) &= \nabla^{-5}f(\zeta+m) - \nabla^{-5}f(\zeta-1) - \frac{(m+1)^{(1)}}{1!} \nabla^{-4}f(\zeta-1) \\ &\quad - \frac{(m+2)^{(2)}}{2!} \nabla^{-3}f(\zeta-1) - \frac{(m+3)^{(3)}}{3!} \nabla^{-2}f(\zeta-1) \\ &\quad - \frac{(m+4)^{(4)}}{4!} \nabla^{-1}f(\zeta-1) \end{aligned}$$

Applying $f(\zeta) = \zeta^{(2)}$, $\zeta = 2$ and $m = 3$ in the above equation, we get

$$\begin{aligned} \text{LHS} &= \sum_{r=0}^m \frac{(r+4)^{(4!)}}{2} f(\zeta + m - r) = (\zeta + 3)^{(2)} + 5(\zeta + 2)^{(2)} + 15(\zeta + 1)^{(2)} + 35\zeta^{(2)} \\ &= 5^{(2)} + (5)4^{(2)} + (15)3^{(2)} + (35)2^{(2)} = 5.4 + 5(4.3) + 15(3.2) + 35(2.1) = 240. \end{aligned}$$

We know that, $\nabla^{-1}\zeta^{(2)} = \frac{(\zeta + 1)^{(3)}}{3}$, $\nabla^{-2}\zeta^{(2)} = \frac{(\zeta + 2)^{(4)}}{12}$, $\nabla^{-3}\zeta^{(2)} = \frac{(\zeta + 3)^{(5)}}{60}$,
 $\nabla^{-4}\zeta^{(2)} = \frac{(\zeta + 4)^{(6)}}{360}$ and $\nabla^{-5}\zeta^{(2)} = \frac{(\zeta + 5)^{(7)}}{2520}$.

$$\begin{aligned} \text{RHS} &= \frac{(\zeta + 5 + 3)^{(7)}}{2520} - \frac{(\zeta + 5 - 1)^{(7)}}{2520} - (4) \frac{(\zeta + 4 - 1)^{(6)}}{360} - (10) \frac{(\zeta + 3 - 1)^{(5)}}{60} - \\ & (20) \frac{(\zeta + 2 - 1)^{(4)}}{12} - (35) \frac{(\zeta + 1 - 1)^{(3)}}{3} \\ &= \frac{(10)^{(7)}}{2520} - \frac{(6)^{(7)}}{2520} - (4) \frac{(5)^{(6)}}{360} - (10) \frac{(4)^{(5)}}{60} - (20) \frac{(3)^{(4)}}{12} - (35) \frac{(2)^{(3)}}{3} = 240. \end{aligned}$$

Therefore, LHS=RHS=240.

Thus we have obtained Nabla operator its inverse and some basic results using these operator.

5 Nabla- ℓ Operator

Definition 5.1 Let f be a real valued function and $\zeta \neq 0$. The Nabla- ℓ operator on f is defined as

$$\nabla_{\ell} f(\zeta) = f(\zeta) - f(\zeta - \ell). \tag{80}$$

The inverse of Nabla- ℓ operator on f is defined by, if there exists $g(\zeta)$ such that

$$\nabla_{\ell} g(\zeta) = f(\zeta) \Leftrightarrow g(\zeta) + c = \nabla_{\ell}^{-1} f(\zeta), \tag{81}$$

where c is an arbitrary constant.

Theorem 5.2 If f be an real valued function and $\nabla_{\ell}^{-1} f(\zeta)$ exists, then

$$\sum_{r=0}^m f(\zeta + (m - r)\ell) = \nabla_{\ell}^{-1} f(\zeta + m\ell) - \nabla_{\ell}^{-1} f(\zeta - \ell), \forall m \in \mathbb{Z}^+. \tag{82}$$

Proof: From equation (80), $f(\zeta) = f(\zeta) - f(\zeta - \ell)$

Consider $\nabla_{\ell}^{-1} f(\zeta) = g(\zeta)$, then $f(\zeta) = \nabla_{\ell} g(\zeta)$

Since $f(\zeta) = g(\zeta) - g(\zeta - \ell)$,

$$f(\zeta) + g(\zeta - \ell) = g(\zeta) \tag{83}$$

Replace ζ by $\zeta + \ell$ in equation (83),

$$f(\zeta + \ell) + g(\zeta) = g(\zeta + \ell)$$

Substitute equation (83),

$$f(\zeta + \ell) + f(\zeta) + g(\zeta - \ell) = g(\zeta + \ell).$$

Again replace ζ by $\zeta + \ell$,

$$f(\zeta + 2\ell) + f(\zeta + \ell) + g(\zeta) = g(\zeta + 2\ell)$$

Substitute equation (60),

$$f(\zeta + 2\ell) + f(\zeta + \ell) + f(\zeta) + g(\zeta - \ell) = g(\zeta + 2\ell).$$

⋮

By induction;

$$f(\zeta + m\ell) + f(\zeta + (m - 1)\ell) + \dots + f(\zeta) + g(\zeta - \ell) = g(\zeta + m\ell)$$

In general, we get

$$f(\zeta) + f(\zeta + \ell) + \dots + f(\zeta + m\ell) = \nabla_{\ell}^{-1} f(\zeta + m\ell) - \nabla_{\ell}^{-1} f(\zeta - \ell)$$

From the above result, we get the equation (82).

Example 5.3 Applying $f(\zeta) = \zeta^{(2)}$, $\zeta = 3$, $\ell = 2$ and $m = 3$ in equation (59), then we have

$$\begin{aligned} \text{LHS} &= \sum_{r=0}^m f(\zeta + (m - r)\ell) = \zeta_{\ell}^{(2)} + (\zeta + 1)_{\ell}^{(2)} + (\zeta + 2)_{\ell}^{(2)} + (\zeta + 3)_{\ell}^{(2)} \\ &= 3_2^{(2)} + 5_2^{(2)} + 7_2^{(2)} + 9_2^{(2)} = 3.1 + 5.3 + 7.5 + 9.7 = 116. \end{aligned}$$

We know that, $\nabla_{\ell}^{-1} \zeta_{\ell}^{(2)} = \frac{(\zeta + 1)_{\ell}^{(3)}}{3\ell}$

$$\begin{aligned} \text{RHS} &= \nabla_{\ell}^{-1} f(\zeta + m\ell) - \nabla_{\ell}^{-1} f(\zeta - \ell) \\ &= \frac{(\zeta + 3\ell + 1)_{\ell}^{(3)}}{3\ell} - \frac{(\zeta + \ell - \ell)_{\ell}^{(3)}}{3\ell} = \frac{(11)_2^{(3)}}{6} - \frac{(3)_2^{(3)}}{6} = \frac{11.9.7.5}{6} - \frac{3.1. - 1}{6} = 116. \end{aligned}$$

Therefore, LHS=RHS=116.

Theorem 5.4 Let f be a real valued function, $\nabla_{\ell}^{-1} f(\zeta)$ and $\nabla_{\ell}^{-2} f(\zeta)$ exists, then

$$\begin{aligned} \sum_{r=0}^m (r + 1)f(\zeta + (m - r)\ell) \\ = \nabla_{\ell}^{-2} f(\zeta + m\ell) - \nabla_{\ell}^{-2} f(\zeta - \ell) - (m + 1)\nabla_{\ell}^{-1} f(\zeta - \ell), \forall m \in Z^+. \end{aligned} \tag{84}$$

Proof: From equation (82), then we get

$$\sum_{r=0}^m f(\zeta + m - r) = \nabla_{\ell}^{-1} f(\zeta + m\ell) - \nabla_{\ell}^{-1} f(\zeta - \ell)$$

Applying ∇_{ℓ}^{-1} on both sides,

$$\nabla_{\ell}^{-1} f(\zeta) + \nabla_{\ell}^{-1} f(\zeta + \ell) + \dots + \nabla_{\ell}^{-1} f(\zeta + m\ell) = \nabla_{\ell}^{-2} f(\zeta + m\ell) - \nabla_{\ell}^{-2} f(\zeta - \ell)$$

Put $m = 4$ in equation (82),

$$f(\zeta) + f(\zeta + \ell) + f(\zeta + 2\ell) + f(\zeta + 3\ell) + f(\zeta + 4\ell) = \nabla_{\ell}^{-1} f(\zeta + 4\ell) - \nabla_{\ell}^{-1} f(\zeta - \ell) \quad (85)$$

Put $m = 3$ in equation (82),

$$f(\zeta) + f(\zeta + \ell) + f(\zeta + 2\ell) + f(\zeta + 3\ell) = \nabla_{\ell}^{-1} f(\zeta + 3\ell) - \nabla_{\ell}^{-1} f(\zeta - \ell) \quad (86)$$

Put $m = 2$ in equation (82),

$$f(\zeta) + f(\zeta + \ell) + f(\zeta + 2\ell) = \nabla_{\ell}^{-1} f(\zeta + 2\ell) - \nabla_{\ell}^{-1} f(\zeta - \ell) \quad (87)$$

Put $m = 1$ in equation (82),

$$f(\zeta) + f(\zeta + \ell) = \nabla_{\ell}^{-1} f(\zeta + \ell) - \nabla_{\ell}^{-1} f(\zeta - \ell) \quad (88)$$

Put $m = 0$ in equation (82),

$$f(\zeta) = \nabla_{\ell}^{-1} f(\zeta) - \nabla_{\ell}^{-1} f(\zeta - \ell) \quad (89)$$

Adding (85) to (89), we get

$$\begin{aligned} 5f(\zeta) + 4f(\zeta + \ell) + 3f(\zeta + 2\ell) + 2f(\zeta + 3\ell) + f(\zeta + 4\ell) \\ = \nabla_{\ell}^{-2} f(\zeta + 4\ell) - \nabla_{\ell}^{-2} f(\zeta - \ell) - (5)\nabla_{\ell}^{-1} f(\zeta - \ell) \end{aligned}$$

In general,

$$\begin{aligned} (m+1)f(\zeta) + mf(\zeta + \ell) + (m-1)f(\zeta + 2\ell) + \dots + (1)f(\zeta + m\ell) \\ = \nabla_{\ell}^{-2} f(\zeta + m\ell) - \nabla_{\ell}^{-2} f(\zeta - \ell) - (m+1)\nabla_{\ell}^{-1} f(\zeta - \ell) \end{aligned}$$

From the above result, we get the equation (84).

Example 5.5 Applying $f(\zeta) = \zeta^{(2)}$, $\zeta = 3$ and $m = 3$ in equation (61), then we have

$$\begin{aligned} \text{LHS} &= \sum_{r=0}^m (r+1)f(\zeta + (m-r)\ell) = (\zeta + 3\ell)_{\ell}^{(2)} + 2(\zeta + 2\ell)_{\ell}^{(2)} + 3(\zeta + \ell)_{\ell}^{(2)} + 4\zeta_{\ell}^{(2)} \\ &= 9_2^{(2)} + (2)7_2^{(2)} + (3)5_2^{(2)} + (4)3_2^{(2)} = 9.7 + 2(7.5) + 3(5/3) + 4(3.1) = 190. \end{aligned}$$

We know that, $\nabla_{\ell}^{-1}\zeta_{\ell}^{(2)} = \frac{(\zeta+1)_{\ell}^{(3)}}{3\ell}$ and $\nabla_{\ell}^{-2}\zeta_{\ell}^{(2)} = \frac{(\zeta+2)_{\ell}^{(4)}}{12\ell^2}$

$$\begin{aligned} \text{RHS} &= \frac{(\zeta+3\ell+2\ell)_{\ell}^{(4)}}{12\ell^2} - \frac{(\zeta+\ell-2\ell)_{\ell}^{(4)}}{12\ell^2} - (4)\frac{(\zeta+\ell-\ell)_{\ell}^{(3)}}{3\ell} \\ &= \frac{(13)_2^{(4)}}{48} - \frac{(5)_2^{(4)}}{48} - (4)\frac{(3)_2^{(3)}}{6} = 190. \end{aligned}$$

Therefore, LHS=RHS=190.

Theorem 5.6 If f be an real valued function and $\nabla_{\ell}^{-1}f(\zeta)$, $\nabla_{\ell}^{-2}f(\zeta)$ and $\nabla_{\ell}^{-3}f(\zeta)$ exists, then

$$\sum_{r=0}^m \frac{(r+2)^{(2)}}{2!} f(\zeta+m-r\ell) = \nabla_{\ell}^{-3}f(\zeta+m\ell) - \nabla_{\ell}^{-3}f(\zeta-\ell) - (m+1)\nabla_{\ell}^{-2}f(\zeta-\ell) - \frac{(m+2)^{(2)}}{2} \nabla_{\ell}^{-1}f(\zeta-\ell), \forall m \in \mathbb{Z}^+. \quad (90)$$

Proof: By equation (84), then we get

$$\sum_{r=0}^m (r+1)f(\zeta+(m-r)\ell) = \nabla_{\ell}^{-2}f(\zeta+m\ell) - \nabla_{\ell}^{-2}f(\zeta-\ell) - (m+1)\nabla_{\ell}^{-1}f(\zeta-\ell)$$

Put $m=4$ in equation (84),

$$\begin{aligned} f(\zeta+4\ell) + 2f(\zeta+3\ell) + 3f(\zeta+2\ell) + 4f(\zeta+\ell) + 5f(\zeta) \\ = \nabla_{\ell}^{-2}f(\zeta+4\ell) - \nabla_{\ell}^{-2}f(\zeta-\ell) - (5)\nabla_{\ell}^{-1}f(\zeta-\ell) \end{aligned} \quad (91)$$

Put $m=3$ in equation (84),

$$f(\zeta+3\ell) + 2f(\zeta+2\ell) + 3f(\zeta+\ell) + 4f(\zeta) = \nabla_{\ell}^{-2}f(\zeta+3\ell) - \nabla_{\ell}^{-2}f(\zeta-\ell) - (4)\nabla_{\ell}^{-1}f(\zeta-\ell) \quad (92)$$

Put $m=2$ in equation (84),

$$f(\zeta+2\ell) + 2f(\zeta+\ell) + 3f(\zeta) = \nabla_{\ell}^{-2}f(\zeta+2\ell) - \nabla_{\ell}^{-2}f(\zeta-\ell) - (3)\nabla_{\ell}^{-1}f(\zeta-\ell) \quad (93)$$

Put $m=1$ in equation (84),

$$f(\zeta+\ell) + 2f(\zeta) = \nabla_{\ell}^{-2}f(\zeta+\ell) - \nabla_{\ell}^{-2}f(\zeta-\ell) - (2)\nabla_{\ell}^{-1}f(\zeta-\ell) \quad (94)$$

Put $m = 0$ in equation (84),

$$f(\zeta) = \nabla_{\ell}^{-2} f(\zeta) - \nabla_{\ell}^{-2} f(\zeta - \ell) - (1) \nabla_{\ell}^{-1} f(\zeta - \ell) \quad (95)$$

Adding (91) to (95), we get

In general,

$$\begin{aligned} & f(\zeta + 4\ell) + (1+2)f(\zeta + 3\ell) + (1+2+3)f(\zeta + 2\ell) + (1+2+3+4)f(\zeta + \ell) \\ & \quad + (1+2+3+4+5)f(\zeta) \\ & = \nabla^{-3} f(\zeta + m) - \nabla_{\ell}^{-3} f(\zeta - \ell) - (m+1) \nabla_{\ell}^{-2} f(\zeta - \ell) - \frac{(6)^{(2)}}{2} \nabla_{\ell}^{-1} f(\zeta - \ell) \end{aligned}$$

From the above result, we get the equation (90).

Example 5.7 Applying $f(\zeta) = \zeta_{\ell}^{(2)}$, $\zeta = 3$, $\ell = 2$ and $m=3$ in Theorem (90), then we have

$$\begin{aligned} \text{LHS} &= \sum_{r=0}^m \frac{(r+2)^{(2)}}{2} f(\zeta + (m-r)\ell) = (\zeta + 3\ell)_{\ell}^{(2)} + 3(\zeta + 2\ell)_{\ell}^{(2)} + 6(\zeta + \ell)_{\ell}^{(2)} + 10\zeta_{\ell}^{(2)} \\ &= 9_2^{(2)} + (3)7_2^{(2)} + (6)5_2^{(2)} + (10)3_2^{(2)} = 9.7 + 3(7.5) + 6(5.3) + 10(3.1) = 288. \end{aligned}$$

$$\text{We know that, } \nabla_{\ell}^{-1} \zeta_{\ell}^{(2)} = \frac{(\zeta + 1)_{\ell}^{(3)}}{3\ell}, \nabla_{\ell}^{-2} \zeta_{\ell}^{(2)} = \frac{(\zeta + 2)_{\ell}^{(4)}}{12\ell^2} \text{ and } \nabla_{\ell}^{-3} \zeta_{\ell}^{(2)} = \frac{(\zeta + 3)_{\ell}^{(5)}}{60\ell^3}.$$

$$\begin{aligned} \text{RHS} &= \nabla^{-3} f(\zeta + m\ell) - \nabla^{-3} f(\zeta - \ell) - (m+1) \nabla^{-2} f(\zeta - \ell) - \frac{(m+2)^{(2)}}{2} \nabla^{-1} f(\zeta - \ell) \\ &= \frac{(\zeta + 3\ell + 3\ell)_{\ell}^{(5)}}{60\ell^3} - \frac{(\zeta + 3\ell - \ell)_{\ell}^{(5)}}{60\ell^2} - (4) \frac{(\zeta + 2\ell - \ell)_{\ell}^{(4)}}{12\ell^2} - (10) \frac{(\zeta + \ell - \ell)_{\ell}^{(3)}}{3\ell} \\ &= \frac{(15)_2^{(5)}}{480} - \frac{(7)_2^{(5)}}{480} - (4) \frac{(5)_2^{(4)}}{48} - (10) \frac{(3)_2^{(3)}}{6} = 288. \end{aligned}$$

Therefore, LHS=RHS=288.

Theorem 5.8 If f be a real valued function and $\nabla_{\ell}^{-1} f(\zeta)$, $\nabla_{\ell}^{-2} f(\zeta)$, $\nabla_{\ell}^{-3} f(\zeta)$ and $\nabla_{\ell}^{-4} f(\zeta)$ exists, then

$$\begin{aligned} \sum_{r=0}^m \frac{(r+3)^{(3)}}{3!} f(\zeta + (m-r)\ell) &= \nabla_{\ell}^{-4} f(\zeta + m\ell) - \nabla_{\ell}^{-4} f(\zeta - \ell) - (m+1) \nabla_{\ell}^{-3} f(\zeta - \ell) \\ & \quad - \frac{(m+2)^{(2)}}{2!} \nabla_{\ell}^{-2} f(\zeta - \ell) \\ & \quad - \frac{(m+3)^{(3)}}{3!} \nabla_{\ell}^{-1} f(\zeta - \ell), \forall m \in \mathbb{Z}^+. \quad (96) \end{aligned}$$

Proof: By equation (90), then we get

$$\sum_{r=0}^m \frac{(r+2)^{(2)}}{2} f(\zeta + (m-r)\ell) = \nabla^{-3} f(\zeta + m\ell) - \nabla_{\ell}^{-3} f(\zeta - \ell) - (m+1) \nabla_{\ell}^{-2} f(\zeta - \ell) - \frac{(m+2)^{(2)}}{2} \nabla_{\ell}^{-1} f(\zeta - \ell)$$

Put $m = 4$ in equation (90),

$$\begin{aligned} \sum_{r=0}^4 \frac{(r+2)^{(2)}}{2} f(\zeta + (4-r)\ell) &= \nabla_{\ell}^{-3} f(\zeta + 4\ell) - \nabla_{\ell}^{-3} f(\zeta - \ell) - (4+1) \nabla_{\ell}^{-2} f(\zeta - \ell) \\ &\quad - \frac{(4+2)^{(2)}}{2} \nabla_{\ell}^{-1} f(\zeta - \ell) \\ \frac{(2)^{(2)}}{2} f(\zeta + 4\ell) + \frac{(3)^{(2)}}{2} f(\zeta + 3\ell) + \frac{(4)^{(2)}}{2} f(\zeta + 2\ell) + \frac{(5)^{(2)}}{2} f(\zeta + \ell) + \frac{(6)^{(2)}}{2} f(\zeta) \\ &= \nabla^{-3} f(\zeta + 4\ell) - \nabla_{\ell}^{-3} f(\zeta - \ell) - (5) \nabla_{\ell}^{-2} f(\zeta - \ell) - \frac{(6)^{(2)}}{2} \nabla_{\ell}^{-1} f(\zeta - \ell) \\ &\quad f(\zeta + 4\ell) + (3)f(\zeta + 3\ell) + (6)f(\zeta + 2\ell) + (10)f(\zeta + \ell) + (15)f(\zeta) \\ &= \nabla_{\ell}^{-3} f(\zeta + 4\ell) - \nabla_{\ell}^{-3} f(\zeta - \ell) - (5) \nabla_{\ell}^{-2} f(\zeta - \ell) - (15) \nabla_{\ell}^{-1} f(\zeta - \ell) \quad (97) \end{aligned}$$

Put $m = 3$ in equation (90),

$$\begin{aligned} \sum_{r=0}^3 \frac{(r+2)^{(2)}}{2} f(\zeta + (3-r)\ell) &= \nabla_{\ell}^{-3} f(\zeta + 3\ell) - \nabla_{\ell}^{-3} f(\zeta - \ell) - (3+1) \nabla_{\ell}^{-2} f(\zeta - \ell) \\ &\quad - \frac{(3+2)^{(2)}}{2} \nabla_{\ell}^{-1} f(\zeta - \ell) \\ \frac{(2)^{(2)}}{2} f(\zeta + 3\ell) + \frac{(3)^{(2)}}{2} f(\zeta + 2\ell) + \frac{(4)^{(2)}}{2} f(\zeta + \ell) + \frac{(5)^{(2)}}{2} f(\zeta) \\ &= \nabla_{\ell}^{-3} f(\zeta + 3\ell) - \nabla_{\ell}^{-3} f(\zeta - \ell) - (4) \nabla_{\ell}^{-2} f(\zeta - \ell) - \frac{(5)^{(2)}}{2} \nabla_{\ell}^{-1} f(\zeta - \ell) \\ &\quad f(\zeta + 3\ell) + (3)f(\zeta + 2\ell) + (6)f(\zeta + \ell) + (10)f(\zeta) \\ &= \nabla_{\ell}^{-3} f(\zeta + 3\ell) - \nabla_{\ell}^{-3} f(\zeta - \ell) - (4) \nabla_{\ell}^{-2} f(\zeta - \ell) - (10) \nabla_{\ell}^{-1} f(\zeta - \ell) \quad (98) \end{aligned}$$

Put $m = 2$ in equation (90),

$$\begin{aligned} \sum_{r=0}^2 \frac{(r+2)^{(2)}}{2} f(\zeta + (2-r)\ell) &= \nabla_{\ell}^{-3} f(\zeta + 2\ell) - \nabla_{\ell}^{-3} f(\zeta - \ell) - (2+1) \nabla_{\ell}^{-2} f(\zeta - \ell) \\ &\quad - \frac{(2+2)^{(2)}}{2} \nabla_{\ell}^{-1} f(\zeta - \ell) \\ \frac{(2)^{(2)}}{2} f(\zeta + 2\ell) + \frac{(3)^{(2)}}{2} f(\zeta + \ell) + \frac{(4)^{(2)}}{2} f(\zeta) &= \nabla_{\ell}^{-3} f(\zeta + 2\ell) - \nabla_{\ell}^{-3} f(\zeta - \ell) \\ &\quad - (3) \nabla_{\ell}^{-2} f(\zeta - \ell) \end{aligned}$$

$$\begin{aligned}
 & f(\zeta + 2\ell) + (3)f(\zeta + \ell) + (6)f(\zeta) - \frac{(4)^{(2)}}{2} \nabla_{\ell}^{-1} f(\zeta - \ell) \\
 & = \nabla_{\ell}^{-3} f(\zeta + 2\ell) - \nabla_{\ell}^{-3} f(\zeta - \ell) - (3)\nabla_{\ell}^{-2} f(\zeta - \ell) - (6)\nabla_{\ell}^{-1} f(\zeta - \ell) \quad (99)
 \end{aligned}$$

Put $m = 1$ in equation (90),

$$\begin{aligned}
 \sum_{r=0}^1 \frac{(r+2)^{(2)}}{2} f(\zeta + \ell) & = \nabla_{\ell}^{-3} f(\zeta + \ell) - \nabla_{\ell}^{-3} f(\zeta - \ell) - (1+1)\nabla_{\ell}^{-2} f(\zeta - \ell) \\
 & \quad - \frac{(1+2)^{(2)}}{2} \nabla_{\ell}^{-1} f(\zeta - \ell) \\
 \frac{(2)^{(2)}}{2} f(\zeta + \ell) + \frac{(3)^{(2)}}{2} f(\zeta) & = \nabla_{\ell}^{-3} f(\zeta + \ell) - \nabla_{\ell}^{-3} f(\zeta - \ell) - (2)\nabla_{\ell}^{-2} f(\zeta - \ell) \\
 & \quad - \frac{(3)^{(2)}}{2} \nabla_{\ell}^{-1} f(\zeta - \ell)
 \end{aligned}$$

$$f(\zeta + \ell) + (3)f(\zeta) = \nabla_{\ell}^{-3} f(\zeta + \ell) - \nabla_{\ell}^{-3} f(\zeta - \ell) - (2)\nabla_{\ell}^{-2} f(\zeta - \ell) - (3)\nabla_{\ell}^{-1} f(\zeta - \ell) \quad (100)$$

Put $m = 0$ in equation (90),

$$\begin{aligned}
 \frac{(2)^{(2)}}{2} f(\zeta) & = \nabla_{\ell}^{-3} f(\zeta) - \nabla_{\ell}^{-3} f(\zeta - \ell) - (1)\nabla_{\ell}^{-2} f(\zeta - \ell) - \frac{(2)^{(2)}}{2} \nabla_{\ell}^{-1} f(\zeta - \ell) \\
 f(\zeta) & = \nabla_{\ell}^{-3} f(\zeta) - \nabla_{\ell}^{-3} f(\zeta - \ell) - \nabla_{\ell}^{-2} f(\zeta - \ell) - \nabla_{\ell}^{-1} f(\zeta - \ell) \quad (101)
 \end{aligned}$$

Adding (97) to (101), we get

$$\begin{aligned}
 f(\zeta + 4\ell) + (1+3)f(\zeta + 3\ell) + (1+3+6)f(\zeta + 2\ell) + (1+3+6+10)f(\zeta + \ell) \\
 + (1+3+6+10+15)f(\zeta) \\
 = \nabla_{\ell}^{-4} f(\zeta + m\ell) - \nabla_{\ell}^{-4} f(\zeta - \ell) - (m+1)\nabla_{\ell}^{-3} f(\zeta - \ell) \\
 - (1+2+3+4+5)\nabla_{\ell}^{-2} f(\zeta - \ell) \\
 - (1+3+6+10+15)\nabla_{\ell}^{-1} f(\zeta - \ell)
 \end{aligned}$$

In general,

$$\begin{aligned}
 f(\zeta + 4\ell) + (4)f(\zeta + 3\ell) + (10)f(\zeta + 2\ell) + (20)f(\zeta + \ell) + (35)f(\zeta) \\
 = \nabla_{\ell}^{-4} f(\zeta + m\ell) - \nabla_{\ell}^{-4} f(\zeta - \ell) - (m+1)\nabla_{\ell}^{-3} f(\zeta - \ell) \\
 - (15)\nabla_{\ell}^{-2} f(\zeta - \ell) - (35)\nabla_{\ell}^{-1} f(\zeta - \ell)
 \end{aligned}$$

From the above result, we get the equation (96).

Example 5.9 Applying $f(\zeta) = \zeta^{(2)}$, $\zeta = 3$ and $m = 3$ in equation (73), then we have

$$\begin{aligned} \text{LHS} &= \sum_{r=0}^m \frac{(r+3)^{(3)}}{3!} f(\zeta + (m-r)\ell) = (\zeta + 3\ell)_\ell^{(2)} + 4(\zeta + 2\ell)_\ell^{(2)} + 10(\zeta + \ell)_\ell^{(2)} + 20\zeta_\ell^{(2)} \\ &= 9_2^{(2)} + (4)7_2^{(2)} + (10)5_2^{(2)} + (20)3_2^{(2)} = 9.7 + 4(7.5) + 10(5.3) + 20(3.1) = 413. \end{aligned}$$

We know that, $\nabla_\ell^{-1}\zeta_\ell^{(2)} = \frac{(\zeta+1)_\ell^{(3)}}{3\ell}$, $\nabla_\ell^{-2}\zeta_\ell^{(2)} = \frac{(\zeta+2)_\ell^{(4)}}{12\ell^2}$, $\nabla_\ell^{-3}\zeta_\ell^{(2)} = \frac{(\zeta+3)_\ell^{(5)}}{60\ell^3}$

and $\nabla_\ell^{-4}\zeta_\ell^{(2)} = \frac{(\zeta+4)_\ell^{(6)}}{360\ell^4}$.

$$\begin{aligned} \text{RHS} &= \nabla_\ell^{-4}f(\zeta + m\ell) - \nabla_\ell^{-4}f(\zeta - \ell) - (m+1)\nabla_\ell^{-3}f(\zeta - \ell) - \frac{(m+2)^{(2)}}{2!}\nabla_\ell^{-2}f(\zeta - \ell) \\ &\quad - \frac{(m+3)^{(3)}}{3!}\nabla_\ell^{-1}f(\zeta - \ell) \\ &= \frac{(\zeta + 4\ell + 3\ell)_\ell^{(6)}}{360\ell^4} - \frac{(\zeta + 4\ell - \ell)_\ell^{(6)}}{360\ell^4} - (4)\frac{(\zeta + 3\ell - \ell)_\ell^{(5)}}{60\ell^3} - (10)\frac{(\zeta + 2\ell - \ell)_\ell^{(4)}}{12\ell^2} \\ &\quad - (20)\frac{(\zeta + \ell - \ell)_\ell^{(3)}}{3\ell} \\ &= \frac{(17)_2^{(6)}}{5760} - \frac{(9)_2^{(6)}}{5760} - (4)\frac{(7)_2^{(5)}}{480} - (10)\frac{(5)_2^{(4)}}{48} - (20)\frac{(3)_2^{(3)}}{6} = 413. \end{aligned}$$

Therefore, LHS=RHS=413.

Theorem 5.10 If f be an real valued function and $\nabla_\ell^{-r}f(\zeta)$ where $r = 1, 2, \dots, n$ exists, then

$$\begin{aligned} \sum_{r=0}^m \frac{(r+n-1)^{(n-1)}}{(n-1)!} f(\zeta + (m-r)\ell) &= \nabla_\ell^{-n}f(\zeta + m\ell) - \nabla_\ell^{-n}f(\zeta - \ell) \\ &\quad - \frac{(m+1)^{(1)}}{1!}\nabla_\ell^{-(n-1)}f(\zeta - \ell) \\ &\quad - \frac{(m+2)^{(2)}}{2!}\nabla_\ell^{-(n-2)}f(\zeta - \ell) \\ &\quad - \frac{(m+n-1)^{(n-1)}}{(n-1)!}\nabla_\ell^{-1}f(\zeta - \ell), \forall m \in \mathbb{Z}^+. \end{aligned} \tag{102}$$

Proof: The proof follows from, put $n = 1$ in Theorem (102),

$$\sum_{r=0}^m f(\zeta + (m-r)\ell) = \nabla_{\ell}^{-1} f(\zeta + m\ell) - \nabla_{\ell}^{-1} f(\zeta - \ell)$$

Put $n = 2$ in Theorem (102),

$$\sum_{r=0}^m (r+1)f(\zeta + (m-r)\ell) = \nabla_{\ell}^{-2} f(\zeta + m\ell) - \nabla_{\ell}^{-2} f(\zeta - \ell) - (m+1)\nabla_{\ell}^{-1} f(\zeta - \ell)$$

Put $n = 3$ in Theorem (102),

$$\begin{aligned} \sum_{r=0}^m \frac{(r+2)^{(2)}}{2!} f(\zeta + (m-r)\ell) &= \nabla_{\ell}^{-3} f(\zeta + m\ell) - \nabla_{\ell}^{-3} f(\zeta - \ell) - (m+1)\nabla_{\ell}^{-2} f(\zeta - \ell) \\ &\quad - \frac{(m+2)^{(2)}}{2} \nabla_{\ell}^{-1} f(\zeta - \ell) \end{aligned}$$

Put $n = 4$ in Theorem (102),

$$\begin{aligned} \sum_{r=0}^m \frac{(r+3)^{(3)}}{3!} f(\zeta + (m-r)\ell) &= \nabla_{\ell}^{-4} f(\zeta + m\ell) - \nabla_{\ell}^{-4} f(\zeta - \ell) - (m+1)\nabla_{\ell}^{-3} f(\zeta - \ell) \\ &\quad - \frac{(m+2)^{(2)}}{2!} \nabla_{\ell}^{-2} f(\zeta - \ell) \\ &\quad - \frac{(m+3)^{(3)}}{3!} \nabla_{\ell}^{-1} f(\zeta - \ell). \end{aligned}$$

Example 5.11 Applying $n=5$ by equation (79)

$$\begin{aligned} \sum_{r=0}^m \frac{(r+4)^{(4)}}{4!} f(\zeta + (m-r)\ell) &= \nabla_{\ell}^{-5} f(\zeta + m\ell) - \nabla_{\ell}^{-5} f(\zeta - \ell) \\ &\quad - \frac{(m+1)^{(1)}}{1!} \nabla_{\ell}^{-4} f(\zeta - \ell) - \frac{(m+2)^{(2)}}{2!} \nabla_{\ell}^{-3} f(\zeta - \ell) \\ &\quad - \frac{(m+3)^{(3)}}{3!} \nabla_{\ell}^{-2} f(\zeta - \ell) - \frac{(m+4)^{(4)}}{4!} \nabla_{\ell}^{-1} f(\zeta - \ell) \end{aligned}$$

Put $f(\zeta) = \zeta_{\ell}^{(2)}$, $\zeta = 2$, $\ell = 2$ and $m = 3$ in the above equation, we get

$$\begin{aligned} \text{LHS} &= \sum_{r=0}^m \frac{(r+4)^{(4)}}{4!} f(\zeta + (m-r)\ell) = (\zeta + 3\ell)_{\ell}^{(2)} + 5(\zeta + 2\ell)_{\ell}^{(2)} + 15(\zeta + \ell)_{\ell}^{(2)} + 35\zeta_{\ell}^{(2)} \\ &= 9_2^{(2)} + (5)7_2^{(2)} + (15)5_2^{(2)} + (35)3_2^{(2)} = 9.7 + 5(7.5) + 15(5.3) + 35(3.1) = 568. \end{aligned}$$

We know that, $\nabla_{\ell}^{-1} \zeta_{\ell}^{(2)} = \frac{(\zeta+1)_{\ell}^{(3)}}{3\ell}$, $\nabla_{\ell}^{-2} \zeta_{\ell}^{(2)} = \frac{(\zeta+2)_{\ell}^{(4)}}{12\ell^2}$, $\nabla_{\ell}^{-3} \zeta_{\ell}^{(2)} = \frac{(\zeta+3)_{\ell}^{(5)}}{60\ell^3}$,

$$\nabla_{\ell}^{-4} \zeta_{\ell}^{(2)} = \frac{(\zeta+4)_{\ell}^{(6)}}{360\ell^4} \text{ and } \nabla_{\ell}^{-5} \zeta_{\ell}^{(2)} = \frac{(\zeta+5)_{\ell}^{(7)}}{2520\ell^5}.$$

$$\begin{aligned} \text{RHS} &= \nabla_{\ell}^{-5} f(\zeta + m\ell) - \nabla_{\ell}^{-5} f(\zeta - \ell) - \frac{(m+1)^{(1)}}{1!} \nabla_{\ell}^{-4} f(\zeta - \ell) \\ &\quad - \frac{(m+2)^{(2)}}{2!} \nabla_{\ell}^{-3} f(\zeta - \ell) - \frac{(m+3)^{(3)}}{3!} \nabla_{\ell}^{-2} f(\zeta - \ell) - \frac{(m+4)^{(4)}}{4!} \nabla_{\ell}^{-1} f(\zeta - \ell) \end{aligned}$$

$$\begin{aligned}
 &= \frac{(\zeta + 5\ell + 3\ell)^{(7)}}{2520\ell^5} - \frac{(\zeta + 5\ell - \ell)^{(7)}}{2520\ell^5} - (4)\frac{(\zeta + 4\ell - \ell)^{(6)}}{360\ell^4} - (10)\frac{(\zeta + 3\ell - \ell)^{(5)}}{60\ell^3} \\
 &\quad - (20)\frac{(\zeta + 2\ell - \ell)^{(4)}}{12\ell^2} - (35)\frac{(\zeta + \ell - \ell)^{(3)}}{3\ell} \\
 &= \frac{(19)^{(7)}}{80640} - \frac{(11)^{(7)}}{80640} - (4)\frac{(9)^{(6)}}{5760} - (10)\frac{(7)^{(5)}}{480} - (20)\frac{(5)^{(4)}}{48} - (35)\frac{(3)^{(3)}}{6} = 568.
 \end{aligned}$$

Therefore, LHS=RHS =568.

Thus we have obtained Nabla- ℓ operators its inverse and some basic results using these operators.

6 Conclusion

In this research work, we developed the theory of discrete version of fundamental theorems using Delta, Delta- ℓ , Nabla and Nabla- ℓ operators. This theory is subsequently applied to establish several higher order fundamental theorems. Our findings are validated with suitable examples in the finds of discrete fractional calculus.

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