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A Study of Generalized Alpa(q) Difference Operator and Its Sum

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Abstract

This paper aims to the generaized $\alpha(q)$- difference operator and its sums. By equating the closed and summation forms of inverse operators, we derive novel summation formulae. These advancements contribute to a deeper understanding of the calculus framework and its applications across various mathematical and scientific fields. This results can be applied to improve dicrete dynamic systems, enchance computational techniques and provide new insights into the behaviour of discrete functions.

Key words: q- Delta operator, $q(\alpha)$- Delta operator, q - Nabla Operator, $q(\alpha)$ - Nabla Operator.

AMS classification: 03E72.

1 Introduction

A difference equation is a mathematical equation that expresses the relationship between successive terms of a sequence. It is the discrete counterpart of a differential equation and is commonly used in fields like economics, engineering, and physics to model discrete-time systems. In discrete mathematics and numerical analysis, difference operators play a fundamental role in studying sequences, discrete functions, and numerical approximations. They provide discrete analogs of differentiation in calculus and are used in solving difference equations, numerical differentiation, and interpolation problems. The two most commonly used difference operators are the Delta operator (Δ) and the Nabla operator (∇). These operators help analyze changes in sequences and discrete functions by evaluating the difference between successive or preceding terms.

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Difference equations play a crucial role in the study of discrete dynamical systems, numerical analysis, and various applied fields, including engineering, economics, and biological sciences. As the discrete counterpart of differential equations, they provide a mathematical framework for modeling and analyzing systems that evolve in discrete time steps. Central to the formulation and solution of difference equations is the Delta operator (Δ), which serves as an essential tool in discrete calculus, interpolation, and numerical methods.

Delta Operator

The Delta operator is analogous to the first derivative in continuous calculus, capturing the rate of change between consecutive values of a function. By recursively applying the Delta operator, higher-order differences can be obtained, which are instrumental in polynomial approximation, interpolation, and solving discrete boundary value problems. In the context of numerical methods, the Delta operator is widely utilized in finite difference approximations of derivatives, particularly in solving differential equations numerically. This formulation provides an algebraic foundation for studying discrete systems and facilitates the development of computational algorithms in numerical analysis.

Due to its significance in mathematical modeling and computational sciences, the Delta operator remains a fundamental concept in difference equations and discrete mathematics. Its applications extend to diverse domains, including signal processing, population dynamics, and financial modeling, where discrete changes over time are analyzed. Theoretical advancements in difference equations continue to enhance the understanding of discrete phenomena, bridging the gap between discrete and continuous mathematical frameworks .Difference equations play a crucial role in the study of discrete dynamical systems, numerical analysis, and various applied fields, including engineering, economics, and biological sciences. As the discrete counterpart of differential equations, they provide a mathematical framework for modeling and analyzing systems that evolve in discrete time steps. Central to the formulation and solution of difference equations is the Delta operator (Δ), which serves as an essential tool in discrete calculus, interpolation, and numerical methods.

Nabla Operator

A key operator in the study of discrete changes within such systems is the Nabla operator (∇), commonly referred to as the Nabla operator. This operator provides a discrete analog to differentiation, particularly in contexts where past values influence the evolution of a function. The Nabla operator is widely applied in areas such as backward difference interpolation, numerical solutions of differential equations, time series analysis, and stochastic processes. Its significance extends across various scientific disciplines, including engineering, economics, and biological modeling, where past data is utilized to predict future behavior. It is used to define and solve difference equations, which are essential in modeling population growth, chemical reactions. The Nabla Operator is a fundamental tool in difference equations, discrete calculus, and time series analysis. It has numerous applications in Mathematics, physics, engineering, and computer Science.

Fibonacci Sequence

The properties of the Fibonacci sequence, particularly its first-order representation, using the nabla operator (also known as the backward difference operator). The nabla operator, denoted by ∇, is defined as $\nabla f(n) = f(n) - f(n-1)$. Using this operator, we can express the rate of change of the Fibonacci sequence backward. The use of nabla calculus allows for an alternative perspective to exploring the Fibonacci sequence, its properties, and to develop solutions with its backward difference and anti-difference formulas. Also, it allows to model scenarios where the analysis and prediction of trends from past observations are desired. This has a great potential to discover new results and applications for the Fibonacci numbers.

The Fibonacci sequence is a sequence of non-negative integers starting with the integer pair 0 and 1, where $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. The first few Fibonacci numbers are: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, $\cdots$. The Fibonacci sequence is perhaps one of the most well-known sequence and it has many interesting properties and important applications to diverse disciplines such as Mathematics, Statistics, Biology, Physics, Finance, Architecture, Computer Science, etc.

2 q-Delta Operator

In this Section, we derive generalized delta operator with shift value q and gives summation formulae related to these operator and also we derive some numerical examples.

Definition 2.1 Let $f(t)$ be a real valued function and t, q be a fixed real numbers. Then the q-Delta operator Δ_q is defined as

$$\Delta_q f(t) = f(tq) - f(t) \tag{1}$$

and the inverse of the q-Delta operator Δ_q^{-1} is defined as

$$f(t) = \Delta_q g(t) \implies \Delta_q^{-1} f(t) = g(t) + c$$

where c is an arbitrary constant.

Theorem 2.2 If f be a real valued function and $\Delta_q^{-1} f(t)$ exists, then for every positive integer m,

$$f(tq^m) + f(tq^{m-1}) + f(tq^{m-2}) + \cdots + f(t) = \Delta_q^{-1} f(tq^{m+1}) - \Delta_q^{-1} f(t). \quad (2)$$

Proof: From equation (2), we have

$$\Delta_q f(t) = f(tq) - f(t)$$

Let $\Delta_q^{-1} f(t) = g(t)$

$$\implies f(t) = \Delta_q g(t)$$

$$f(t) + g(t) = g(tq) \quad (3)$$

Replace t by tq in the above equation,

$$f(tq) + g(tq) = g(tq^2)$$

Substitute (3) in above equation,

$$f(tq) + f(t) + g(t) = g(tq^2)$$

Again replace t by tq in the above equation,

$$f(tq^2) + f(tq) + g(tq) = g(tq^3)$$

Substitute (3) in above equation,

$$f(tq^2) + f(tq) + f(t) + g(t) = g(tq^3)$$

In general,

$f(tq^m) + f(tq^{m-1}) + f(tq^{m-2}) + \cdots + f(t) + g(t) = g(tq^{m+1})$

$$\therefore f(tq^m) + f(tq^{m-1}) + f(tq^{m-2}) + \cdots + f(t) = \Delta_q^{-1} f(tq^{m+1}) - \Delta_q^{-1} f(t). \quad (4)$$

Example 2.3 Applying $f(t) = t$, $t = 2$, $m = 2$, $q = 3$ in the equation (4), we get

$\boldsymbol{LHS} = f(tq^m) + f(tq^{m-1}) + f(tq^{m-2}) + \cdots + f(t)$

$= (tq^m) + (tq^{m-1}) + (tq^{m-2})$

$= (2)(3)^2 + (2)(3)^1 + (2) = 26.$

We know that, $\Delta_q f(t) = f(tq) - f(t)$

$$\Delta_{\alpha(q)}(t) = (tq) - (t) \implies \Delta_q^{-1} t = \frac{t}{(q-1)}$$

RHS$= \Delta_q^{-1} f(tq^{m+1}) - \Delta_q^{-1} f(t)$

$$= \left(\frac{tq^{m+1}}{q-1}\right) - \left(\frac{t}{(q-1)}\right)$$

$$= \left(\frac{(2)(3)^3}{3-1}\right) - \left(\frac{2}{(3-1)}\right) = 26.$$

$$\therefore LHS = RHS$$

Theorem 2.4 If f be real valued function and $\Delta_q^{-r} f(t)$, where $r = 1, 2$ exists, then for every positive integer m,

$$f(tq^m) + 2f(tq^{m-1}) + 3f(tq^{m-2}) + \cdots + (m+1)f(t) = \Delta_q^{-2} f(tq^{m+2}) - \Delta_q^{-2} f(t) - (m+2)\Delta_q^{-1} f(t).$$

Proof: Consider,

$$f(tq^m) + f(tq^{m-1}) + f(tq^{m-2}) + \cdots + f(t) = \Delta_q^{-1} f(tq^{m+1}) - \Delta_q^{-1} f(t) \quad (5)$$

Applying Δ_q^{-1} in the equation (2),

$$\Delta_q^{-1} f(tq^m) + \Delta_q^{-1} f(tq^{m-1}) + \cdots + \Delta_q^{-1} f(t) = \Delta_q^{-2} f(tq^{m+1}) - \Delta_q^{-2} f(t)$$

Suppose $m = 4, 3, 2, 1$ and 0 in the equation (5), we get

$$f(tq^4) + f(tq^3) + f(tq^2) + f(tq) + f(t) = \Delta_q^{-1} f(tq^5) - \Delta_q^{-1} f(t) \quad (6)$$

$$f(tq^3) + f(tq^2) + f(tq) + f(t) = \Delta_q^{-1} f(tq^4) - \Delta_q^{-1} f(t) \quad (7)$$

$$f(tq^2) + f(tq) + f(t) = \Delta_q^{-1} f(tq^3) - \Delta_q^{-1} f(t) \quad (8)$$

$$f(tq^1) + f(t) = \Delta_q^{-1} f(tq^2) - \Delta_q^{-1} f(t) \quad (9)$$

$$f(t) = \Delta_q^{-1} f(tq) - \Delta_q^{-1} f(t) \quad (10)$$

$$0 = \Delta_q^{-1} f(t) - \Delta_q^{-1} f(t) \quad (11)$$

Adding the equations (6) to (11), we get

$f(tq^4) + 2f(tq^3) + 3f(tq^2) + 4f(tq) + 5f(t) = \Delta_q^{-2} f(tq^6) - \Delta_q^{-2} f(t) - 6\Delta_q^{-1} f(t)$

In general,

$$\therefore f(tq^m)+2f(tq^{m-1})+3f(tq^{m-2})+\cdots+(m+1)f(t)$$
$$=\Delta_q^{-2}f(tq^{m+2})-\Delta_q^{-2}f(t)-(m+2)\Delta_q^{-1}f(t). \quad (12)$$

Example 2.5 Applying $f(t)=t$, $t=3$, $q=4$, $m=2$ in the equation (12), we get
$\boldsymbol{LHS}= f(tq^m)+2f(tq^{m-1})+3f(tq^{m-2})+\cdots+(m+1)f(t)$
$=(tq^m)+2(tq^{m-1})+3(tq^{m-2})$
$=(3)(4^2)+2[(3)(4)]+(3)(3)=81.$
We know that, $\Delta_q f(t)=f(tq)-f(t)$
$\Delta_q^{-1}t=\left(\frac{t}{q-1}\right)$
$\Delta_q^{-2}t=\left(\frac{t}{(q-1)^2}\right)$
$\boldsymbol{RHS}= \Delta_q^{-2}f(tq^{m+2})-\Delta_q^{-2}f(t)-(m+2)\Delta_q^{-1}f(t)$
$=\left(\frac{tq^{m+2}}{(q-1)^2}\right)-\left(\frac{t}{(q-1)^2}\right)-(m+2)\left(\frac{t}{(q-1)}\right)$
$=\left(\frac{(3)(4)^4}{(3^2)}\right)-\left(\frac{3}{3^2}\right)-(4)\left(\frac{3}{3}\right)$
$=81.$
$$\therefore LHS=RHS$$

Theorem 2.6 If f be a real valued function and $\Delta_q^{-r}f(t)$ where $r=1,2,3$ exists, then for every positive integer m,
$$\sum_{r=0}^{m}\left(\frac{(r+2)^{(2)}}{2}\right)f(tq^{m-r})=\Delta_q^{-3}f(tq^{m+3})-\Delta_q^{-3}f(t)-(m+3)\Delta_q^{-2}f(t)-\left(\frac{(m+3)^{(2)}}{2}\right)\Delta_q^{-1}f(t).$$

Proof: Consider,

$$f(tq^m)+2f(tq^{m-1})+\cdots+(m+1)f(t)=\Delta_q^{-2}f(tq^{m+2})-\Delta_q^{-2}f(t)-(m+2)\Delta_q^{-1}f(t) \quad (13)$$

Applying Δ_q^{-1} in the equation (2),
$\Delta_q^{-2}f(tq^m)+\Delta_q^{-2}f(tq^{m-1})+\cdots+\Delta_q^{-2}f(t)=\Delta_q^{-3}f(tq^{m+1})-\Delta_q^{-3}f(t)$

Suppose $m = 4, 3, 2, 1$ and 0 in (13), we get

$$f(tq^4) + 2f(tq^3) + 3f(tq^2) + 4f(t) + 5f(t) = \Delta_q^{-2} f(tq^6) - \Delta_q^{-2} f(t) - (6)\Delta_q^{-1} f(t) \quad (14)$$

$$f(tq^3) + 2f(tq^2) + 3f(tq^2) + 4f(t) = \Delta_q^{-2} f(tq^5) - q^{-2} f(t) - (5)\Delta_q^{-1} f(t) \quad (15)$$

$$f(tq^2) + 2f(tq) + 3f(t) = \Delta_q^{-2} f(tq^4) - \Delta_q^{-2} f(t) - (4)\Delta_q^{-1} f(t) \quad (16)$$

$$f(tq) + 2f(t) = \Delta_q^{-2} f(tq^3) - \Delta_q^{-2} f(t) - (3)\Delta_q^{-1} f(t) \quad (17)$$

$$f(t) = \Delta_q^{-2} f(tq^2) - \Delta_q^{-2} f(t) - (2)\Delta_q^{-1} f(t) \quad (18)$$

$$0 = \Delta_q^{-2} f(tq) - \Delta_q^{-2} f(t) - \Delta_q^{-1} f(t) \quad (19)$$

$$0 = \Delta_q^{-2} f(t) - \Delta_q^{-2} f(t) \quad (20)$$

Adding the equations (14) to (20),we get

$$f(tq^4) + (1+2)f(tq^3) + (1+2+3)f(tq^2) + (1+2+3+4)f(tq) + (1+2+3+4+5)f(t)$$
$$= \Delta_q^{-3} f(tq^7) - \Delta_q^{-3} f(t) - 7\Delta_q^{-2} f(t) - \left(\frac{7^{(2)}}{2}\right)\Delta_q^{-1} f(t)$$

In general,

$$\therefore \sum_{r=0}^{m} \left(\frac{(r+2)^{(2)}}{2}\right) f(tq^{m-r})$$

$$= \Delta_q^{-3} f(tq^{m+3}) - \Delta_q^{-3} f(t) - (m+3)\Delta_q^{-2} f(t) - \left(\frac{(m+3)^{(2)}}{2}\right)\Delta_q^{-1} f(t). \quad (21)$$

Example 2.7 Applying $f(t) = t$, $t = 2$, $m = 3$, $q = 6$ in equation (21), we get

$$\textbf{\textit{LHS}} = \sum_{r=0}^{m} \left(\frac{(r+2)^{(2)}}{2}\right) f(tq^{m-r})$$
$$= \sum_{r=0}^{3} \left(\frac{(r+2)^{(2)}}{2}\right) (tq^{m-r})$$
$$= \left(\frac{2^{(2)}}{2}\right)(2)(6^{3-0}) + \left(\frac{3^{(2)}}{2}\right)(2)(6^{3-1}) + \left(\frac{4^{(2)}}{2}\right)(2)(6^{3-2}) + \left(\frac{5^{(2)}}{2}\right)(2)(6^{3-3})$$
$$= 2268.$$

We know that,

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$\Delta_q^{-1}t = \dfrac{t}{(q-1)}, \ \Delta_q^{-2}t = \dfrac{t}{(q-1)^2}, \ \Delta_q^{-3}t = \dfrac{t}{(q-1)^3}$

$$\boldsymbol{RHS} = \Delta_q^{-3} f(tq^{m+3}) - \Delta_q^{-3} f(t) - (m+3)\Delta_q^{-2} f(t) - \left(\frac{(m+3)^{(2)}}{2}\right)\Delta_q^{-1} f(t)$$
$$= \left(\frac{(2)(6^6)}{(5^3)}\right) - \left(\frac{(2)}{(6-1)^3}\right) - (6)\left(\frac{(2)}{(6-1)^2}\right) - \left(\frac{(6)^{(2)}}{2}\right)\left(\frac{(2)}{(6-1)}\right)$$
$$= 2268$$
$$\therefore \Delta_q^{-3} f(tq^{m+3}) - \Delta_q^{-3} f(t) - (m+3)\Delta_q^{-2} f(t) - \left(\frac{(m+3)^{(2)}}{2}\right)\Delta_q^{-1} f(t) = 2268.$$

Theorem 2.8 If f be a real-valued function and $\Delta_q^{-r} f(t)$ where $r = 1, 2, 3$ exists, then for every positive integer m,

$$\sum_{r=0}^{m}\left(\frac{(r+3)^{(3)}}{3^{(2)}}\right) f(tq^{m-r}) = \Delta_q^{-4} f(tq^{m+4}) - \Delta_q^{-4} f(t) - (m+4)\Delta_q^{-3} f(t). - \left(\frac{(m+4)^{(2)}}{2}\right)\Delta_q^{-2} f(t) - \left(\frac{(m+4)^{(3)}}{3^{(2)}}\right)\Delta_q^{-1} f(t). \quad (22)$$

Example 2.9 Applying $f(t) = t$, $t = 3$, $q = 5$, $m = 2$ in the equation (22), we get

$$\boldsymbol{LHS} = \sum_{r=0}^{m}\left(\frac{(r+3)^{(3)}}{3^{(2)}}\right) f(tq^{m-r})$$
$$= \sum_{r=0}^{2}\left(\frac{(r+3)^{(3)}}{3^{(2)}}\right)(tq^{2-r})$$
$$= \left(\frac{3^{(3)}}{3^{(2)}}\right)(3)(5^2) + \left(\frac{4^{(3)}}{3^{(2)}}\right)(3)(5^1) + \left(\frac{5^{(3)}}{3^{(2)}}\right)(3)$$
$$= 525.$$

We know that, $\Delta_q^{-1}t = \dfrac{t}{(q-1)^1} \ \Delta_q^{-2}t = \dfrac{t}{(q-1)^2},$

$\Delta_q^{-3}t = \dfrac{t}{(q-1)^3}, \ \Delta_q^{-4}t = \dfrac{t}{(q-1)^4}$

$$\boldsymbol{RHS} = \Delta_q^{-4} f(tq^{m+4}) - \Delta_q^{-4} f(t) - (m+4)\Delta_q^{-3} f(t) - \left(\frac{(m+4)^{(2)}}{2}\right)\Delta_q^{-2} f(t) - \left(\frac{(m+4)^{(3)}}{3^{(2)}}\right)\Delta_q^{-1} f(t)$$

$$= \left\{ \left(\frac{(3)(5^6)}{(5-3)^4} \right) - \left(\frac{(3)}{(5-3)^4} \right) - (6) \left(\frac{(3)}{(5-3)^3} \right) - \left(\frac{(6)(5)}{2} \right) \left(\frac{(3)}{(5-3)^2} \right) - \left(\frac{(6)(5)(4)}{(3)(2)} \right) \left(\frac{(3)}{(5-3)^1} \right) \right\}$$

$$= 525.$$

$$\therefore LHS = RHS$$

Theorem 2.10 If f be a real valued function, $\alpha \neq 0$ and $\Delta_{\alpha(q)}^{-n} f(t)$ where $r = 1, 2, \cdots, n$ exists, then for every positive integer m,

$$\sum_{r=0}^{m} \left(\frac{(r+n-1)^{(n-1)}}{(n-1)!} \right) f(tq^{m-r}) = \left\{ \Delta_q^{-n} f(tq^{m+n}) - \Delta_q^{-n} f(t) - \left(\frac{(m+n)^{(1)}}{1!} \right) \Delta_q^{-(n-1)} f(t) - \left(\frac{(m+n)^{(2)}}{2!} \right) \Delta_q^{-(n-2)} f(t) - \cdots - \left(\frac{(m+n)^{(n-1)}}{(n-1)!} \right) \alpha^{m+1} \Delta_{\alpha(q)}^{-1} f(t) \right\} \tag{23}$$

Proof: For $n = 1$ in the equation (23), which gives

$$f(tq^m) + f(tq^{m-1}) + f(tq^{m-2}) + \cdots + f(t) = \Delta_q^{-1} f(tq^{m+1}) - \Delta_q^{-1} f(t)$$

For $n = 2$ in the equation (23), which gives

$$f(tq^m) + f(tq^{m-1}) + \cdots + (m+1) f(t) = \Delta_q^{-2} f(tq^{m+2}) - \Delta_q^{-2} f(t) - (m+2) \Delta_q^{-1} f(t)$$

For $n = 3$ in the equation (23), which gives

$$\sum_{r=0}^{m} \left(\frac{(r+2)^{(2)}}{2} \right) f(tq^{m-r}) = \Delta_q^{-3} f(tq^{m+3}) - \Delta_q^{-3} f(t) - (m+3) \Delta_q^{-2} f(t) - \left(\frac{(m+3)^{(2)}}{2} \right) \Delta_q^{-1} f(t)$$

Proceeding this upto n times, we get

$$\therefore \sum_{r=0}^{m} \left(\frac{(r+n-1)^{(n-1)}}{(n-1)!} \right) f(tq^{m-r}) = \left\{ \Delta_q^{-n} f(tq^{m+n}) - \Delta_q^{-n} f(t) - \left(\frac{(m+n)^{(1)}}{1!} \right) \Delta_q^{-(n-1)} f(t) - \left(\frac{(m+n)^{(2)}}{2!} \right) \Delta_q^{-(n-2)} f(t) - \cdots - \left(\frac{(m+n)^{(n-1)}}{(n-1)!} \right) \alpha^{m+1} \Delta_{\alpha(q)}^{-1} f(t) \right\}.$$

Thus, we have derived generalized theorems of the delta operator with shift value q which gives the summation formula related to these operator and these results are validated with suitable examples.

Definition 2.11 Let f be a real valued function and $\zeta \neq 0$. The Delta operator on f is defined as

$$\Delta f(\zeta) = f(\zeta+1) - f(\zeta). \tag{24}$$

The inverse of Delta operaor on f is defined by, if there exists $g(\zeta)$ such that

$$\Delta g(\zeta) = f(\zeta) \Leftrightarrow g(\zeta) + c = \Delta^{-1} f(\zeta), \tag{25}$$

where c is an arbitrary constant.

Theorem 2.12 If f be a real valued function and $\Delta^{-1} f(\zeta)$

$$\sum_{r=0}^{m} f(\zeta+m-r) = \Delta^{-1} f(\zeta+m+1) - \Delta^{-1} f(\zeta), \forall m \in Z^{+}. \tag{26}$$

Proof: From equation (55), $f(\zeta) = f(\zeta+1) - f(\zeta)$.
Consider $\Delta^{-1} f(\zeta) = g(\zeta)$, then $f(\zeta) = \Delta g(\zeta)$
Since $f(\zeta) = g(\zeta+1) - g(\zeta)$.

$$f(\zeta) + g(\zeta) = g(\zeta+1). \tag{27}$$

Replace ζ by $\zeta+1$ in equation (27),

$$f(\zeta+1) + g(\zeta+1) = g(\zeta+2)$$

Substute equation (27),

$$f(\zeta+1) + f(\zeta) + g(\zeta) = g(\zeta+2).$$

Again replace ζ by $\zeta+1$,

$$f(\zeta+2) + f(\zeta+1) + g(\zeta+1) = g(\zeta+3)$$

Substitute equation (27),

$$f(\zeta+2) + f(\zeta+1) + f(\zeta) + g(\zeta) = g(\zeta+3).$$

$$\vdots$$

By induction;

$$f(\zeta+m)+f(\zeta+m-1)+\cdots+f(\zeta)+g(\zeta)=g(\zeta+m+1)$$
$$f(\zeta)+f(\zeta+1)+\cdots+f(\zeta+m)=\Delta^{-1}f(\zeta+m+1)-\Delta^{-1}f(\zeta)$$

From the above result, we get the equation (110).

Example 2.13 Applying $f(\zeta)=\zeta^{(3)}$, $\zeta=2$ and $m=3$ in equation (110), then we have

$=(\zeta)^{(3)}+(\zeta+1)^{(3)}+(\zeta+2)^{(3)}+(\zeta+3)^{(3)}=(2)^{(3)}+(2+1)^{(3)}+(2+2)^{(3)}+(2+3)^{(3)}$

$$=(2)^{(3)}+(3)^{(3)}+(4)^{(3)}+(5)^{(3)}=0+6+24+60=90.$$

We know that, $\Delta^{-1}\zeta^{(3)}=\dfrac{\zeta^4}{4}$

RHS= $\dfrac{(2+3+1)^{(4)}}{4}-\dfrac{2^{(4)}}{4}=\dfrac{6.5.4.3}{4}-\dfrac{2.1.0.-1}{4}=90.$

Therefore, LHS=RHS=90.

Theorem 2.14 If f be a real valued function, $\Delta^{-1}f(\zeta)$ and $\Delta^{-2}f(\zeta)$ exists, then

$$\sum_{r=0}^{m}(r+1)f(\zeta+m-r)=\Delta^{-2}f(\zeta+m+2)-\Delta^{-2}f(\zeta)-(m+2)\Delta^{-1}f(\zeta),\forall m\in Z^{+}. \tag{28}$$

Proof: By equation (110), then we have

$$\sum_{r=0}^{m}f(\zeta+m-r)=\Delta^{-1}f(\zeta+m+1)-\Delta^{-1}f(\zeta)$$

Applying Δ^{-1} on both sides,

$$\Delta^{-1}f(t)+\Delta^{-1}f(\zeta+1)+\cdots+\Delta^{-1}f(\zeta+m)=\Delta^{-2}f(\zeta+m+1)-\Delta^{-2}f(\zeta)$$

Put $m=3$ in equation (110),

$$f(\zeta)+f(\zeta+1)+f(\zeta+2)+f(\zeta+3)=\Delta^{-1}f(\zeta+4)-\Delta^{-1}f(\zeta) \tag{29}$$

Put $m=2$ in equation (110),

$$f(\zeta)+f(\zeta+1)+f(\zeta+2)=\Delta^{-1}f(\zeta+3)-\Delta^{-1}f(\zeta) \tag{30}$$

Put $m=1$ in equation (110),

$$f(\zeta)+f(\zeta+1)=\Delta^{-1}f(\zeta+2)-\Delta^{-1}f(\zeta) \tag{31}$$

Put $m=0$ in equation (110),

$$f(\zeta)=\Delta^{-1}f(\zeta+1)-\Delta^{-1}f(\zeta) \tag{32}$$

$$0=\Delta^{-1}f(\zeta)-\Delta^{-1}f(\zeta) \tag{33}$$

Adding (84) to (109) , we get

$$4f(\zeta)+3f(\zeta+1)+2f(\zeta+2)+f(\zeta+3)=\Delta^{-2}f(\zeta+5)-\Delta^{-2}f(\zeta)-5\Delta^{-1}f(\zeta)$$

In general,

$$(m+1)f(\zeta)+mf(\zeta+1)+(m-1)f(\zeta+2)+\cdots+(1)f(\zeta+m)$$
$$=\Delta^{-2}f(\zeta+m+2)-\Delta^{-2}f(\zeta)-(m+2)\Delta^{-1}f(\zeta)$$

From the above result, we get the equation (111).

Example 2.15 Applying $f(\zeta)=\zeta^{(3)}$, $\zeta=2$ and m=3 in equation (111), then we have

LHS=$(\zeta+3)^{(3)}+2(\zeta+2)^{(3)}+3(\zeta+1)^{(3)}+4(\zeta)^{(3)}$

$=(2+3)^{(3)}+2(2+2)^{(3)}+3(2+1)^{(3)}+4(2)^{(3)}=60+2(24)+3(6)+0=126.$

We know that, $\Delta^{-1}\zeta^{(3)}=\dfrac{\zeta^4}{4}$ and $\Delta^{-2}\zeta^{(3)}=\dfrac{\zeta^5}{5.4}$

RHS $=\dfrac{(2+3+2)^{(5)}}{5.4}-\dfrac{(2)^{(5)}}{5.4}-(3+2)\dfrac{(2)^{(4)}}{4}=\dfrac{(7)^{(5)}}{5.4}-\dfrac{(2)^{(5)}}{5.4}-(5)\dfrac{(2)^{(4)}}{4}$

$=\dfrac{(7.6.5.4.3)}{5.4}-0-(5)\dfrac{(2.1.0)^{(3)}}{3}=126.$

Therefore, LHS=RHS=126.

Theorem 2.16 If f be a real valued function and $\Delta^{-1}f(\zeta)$, $\Delta^{-2}f(\zeta)$ and $\Delta^{-3}f(\zeta)$ exist, then

$$\sum_{r=0}^{m}\frac{(r+2)^{(2)}}{2}f(\zeta+m-r)=\Delta^{-3}f(\zeta+m+3)-\Delta^{-3}f(\zeta)-(m+3)\Delta^{-2}f(\zeta)$$
$$-\frac{(m+3)^{(2)}}{2}\Delta^{-1}f(\zeta),\forall m\in Z^{+}. \tag{34}$$

Proof: By equation (111), we have

$$\sum_{r=0}^{m}(r+1)f(\zeta+m-r)=\Delta^{-2}f(\zeta+m+2)-\Delta^{-2}f(\zeta)-(m+2)\Delta^{-1}f(\zeta)$$

Put $m = 4$ in equation (111),

$$f(\zeta+4)+2f(\zeta+3)+3f(\zeta+2)+4f(\zeta+1)+5f(\zeta)=\Delta^{-2}f(\zeta+6)-\Delta^{-2}f(\zeta)-(6)\Delta^{-1}f(\zeta) \tag{35}$$

Put $m = 3$ in equation (111),

$$f(\zeta+3)+2f(\zeta+2)+3f(\zeta+1)+4f(\zeta)=\Delta^{-2}f(\zeta+5)-\Delta^{-2}f(\zeta)-(5)\Delta^{-1}f(\zeta) \tag{36}$$

Put $m = 2$ in equation (111),

$$f(\zeta+2)+2f(\zeta+1)+3f(\zeta)=\Delta^{-2}f(\zeta+4)-\Delta^{-2}f(\zeta)-(4)\Delta^{-1}f(\zeta) \tag{37}$$

Put $m = 1$ in equation (111),

$$f(\zeta+1)+2f(\zeta)=\Delta^{-2}f(\zeta+3)-\Delta^{-2}f(\zeta)-(3)\Delta^{-1}f(\zeta) \tag{38}$$

Put $m = 0$ in equation (111),

$$f(\zeta)=\Delta^{-2}f(\zeta+2)-\Delta^{-2}f(\zeta)-(2)\Delta^{-1}f(\zeta) \tag{39}$$

$$0=\Delta^{-2}f(\zeta+1)-\Delta^{-2}f(\zeta)-(1)\Delta^{-1}f(\zeta) \tag{40}$$

$$0=\Delta^{-2}f(\zeta)-\Delta^{-2}f(\zeta)-(0)\Delta^{-1}f(\zeta) \tag{41}$$

Adding (35) to (41), we get
In general,

$$\begin{aligned} f(\zeta+4)+(1+2)f(\zeta+3)+(1+2+3)f(\zeta+2)+(1+2+3+4)f(\zeta+1) \\ +(1+2+3+4+5)f(\zeta) \\ =\Delta^{-3}f(\zeta+m+3)-\Delta^{-3}f(\zeta)-(m+3)\Delta^{-2}f(\zeta)-\frac{(7)^{(2)}}{2}\Delta^{-1}f(\zeta) \end{aligned}$$

From the above result, we get the equation (112).

Example 2.17 Applying $f(\zeta) = \zeta^{(3)}$, $\zeta = 2$ and $m = 3$ in equation (112), then we have

LHS=$\sum_{r=0}^{3} \frac{(r+2)^{(2)}}{2} f(\zeta+3-r)$

$$= \frac{(2)^{(2)}}{2}(2+3)^{(3)} + \frac{(3)^{(2)}}{2}(2+2)^{(3)} + \frac{(4)^{(2)}}{2}(2+1)^{(3)} + \frac{(5)^{(2)}}{2}(2)^{(3)}$$

$$= 5^{(3)} + 3(4)^{(3)} + 6(3)^{(3)} + 10(2)^{(3)} = 60 + 3(24) + 6(6) + 10(0) = 168.$$

We know that, $\Delta^{-1}\zeta^{(3)} = \frac{\zeta^4}{4}$, $\Delta^{-2}\zeta^{(3)} = \frac{\zeta^5}{5.4}$ and $\Delta^{-3}\zeta^{(3)} = \frac{\zeta^6}{6.5.4}$

RHS$= \frac{(2+3+3)^{(6)}}{6.5.4} - \frac{(2)^{(6)}}{6.5.4} - 6\frac{(2)^{(5)}}{5.4} - \frac{(6)^{(2)}}{2}\frac{(2)^{(4)}}{4} = \frac{(8.7.6.5.4.3)}{6.5.4} - 0 - 0 - 0 = 168.$

Therefore, LHS=RHS=168.

Theorem 2.18 If f be a real valued function and $\Delta^{-1}f(\zeta)$, $\Delta^{-2}f(\zeta)$, $\Delta^{-3}f(\zeta)$ and $\Delta^{-4}f(\zeta)$ exists, then

$$\sum_{r=0}^{m}\frac{(r+3)^{(3)}}{3^{(2)}}f(\zeta+m-r) = \Delta^{-4}f(\zeta+m+4) - \Delta^{-4}f(\zeta) - (m+4)\Delta^{-3}f(\zeta) - \frac{(m+4)^{(2)}}{2}\Delta^{-2}f(\zeta) - \frac{(m+4)^{(3)}}{3^{(2)}}\Delta^{-1}f(\zeta), \forall m \in Z^+. \quad (42)$$

Proof: By equation (112), we have

$$\sum_{r=0}^{m}\frac{(r+2)^{(2)}}{2}f(\zeta+m-r) = \Delta^{-3}f(\zeta+m+3) - \Delta^{-3}f(\zeta) - (m+3)\Delta^{-2}f(\zeta) - \frac{(m+3)^{(2)}}{2}\Delta^{-1}f(\zeta)$$

Put $m = 4$ in equation (112),

$$\sum_{r=0}^{3}\frac{(r+2)^{(2)}}{2}f(\zeta+4-r) = \Delta^{-3}f(\zeta+4+3) - \Delta^{-3}f(\zeta) - (4+3)\Delta^{-2}f(\zeta) - \frac{(4+3)^{(2)}}{2}\Delta^{-1}f(\zeta),$$

$$\frac{(2)^{(2)}}{2}f(\zeta+4) + \frac{(3)^{(2)}}{2}f(\zeta+3) + \frac{(4)^{(2)}}{2}f(\zeta+2) + \frac{(5)^{(2)}}{2}f(\zeta+1) + \frac{(6)^{(2)}}{2}f(\zeta) = \Delta^{-3}f(\zeta+6) - \Delta^{-3}f(\zeta) - (6)\Delta^{-2}f(\zeta) - \frac{(6)^{(2)}}{2}\Delta^{-1}f(\zeta)$$

$$f(\zeta+4) + (3)f(\zeta+3) + (6)f(\zeta+2) + (10)f(\zeta+1) + (15)f(\zeta) = \Delta^{-3}f(\zeta+7) - \Delta^{-3}f(\zeta) - (7)\Delta^{-2}f(\zeta) - \frac{(7)^{(2)}}{2}\Delta^{-1}f(\zeta) \quad (43)$$

Put $m = 3$ in equation (112),

$$\sum_{r=0}^{3}\frac{(r+2)^{(2)}}{2}f(\zeta+3-r) = \Delta^{-3}f(\zeta+3+3) - \Delta^{-3}f(\zeta) - (3+3)\Delta^{-2}f(\zeta) - \frac{(3+3)^{(2)}}{2}\Delta^{-1}f(\zeta),$$

$$\frac{(2)^{(2)}}{2}f(\zeta+3)+\frac{(3)^{(2)}}{2}f(\zeta+2)+\frac{(4)^{(2)}}{2}f(\zeta+1)+\frac{(5)^{(2)}}{2}f(\zeta)$$
$$=\Delta^{-3}f(\zeta+6)-\Delta^{-3}f(\zeta)-(6)\Delta^{-2}f(\zeta)-\frac{(6)^{(2)}}{2}\Delta^{-1}f(\zeta)$$
$$f(\zeta+3)+(3)f(\zeta+2)+(6)f(\zeta+1)+(10)f(\zeta)$$
$$=\Delta^{-3}f(\zeta+6)-\Delta^{-3}f(\zeta)-(6)\Delta^{-2}f(\zeta)-\frac{(6)^{(2)}}{2}\Delta^{-1}f(\zeta) \tag{44}$$

Put $m=2$ in equation (112),

$$\sum_{r=0}^{2}\frac{(r+2)^{(2)}}{2}f(\zeta+2-r)=\Delta^{-3}f(\zeta+2+3)-\Delta^{-3}f(\zeta)-(2+3)\Delta^{-2}f(\zeta)-\frac{(2+3)^{(2)}}{2}\Delta^{-1}f(\zeta),$$
$$\frac{(2)^{(2)}}{2}f(\zeta+2)+\frac{(3)^{(2)}}{2}f(\zeta+1)+\frac{(4)^{(2)}}{2}f(\zeta)$$
$$=\Delta^{-3}f(\zeta+5)-\Delta^{-3}f(\zeta)-(5)\Delta^{-2}f(\zeta)-\frac{(5)^{(2)}}{2}\Delta^{-1}f(\zeta)$$
$$f(\zeta+2)+(3)f(\zeta+1)+(6)f(\zeta)=\Delta^{-3}f(\zeta+5)-\Delta^{-3}f(\zeta)-(5)\Delta^{-2}f(\zeta)-\frac{(5)^{(2)}}{2}\Delta^{-1}f(\zeta) \tag{45}$$

Put $m=1$ in equation (112),

$$\sum_{r=0}^{1}\frac{(r+2)^{(2)}}{2}f(\zeta+1-r)=\Delta^{-3}f(\zeta+1+3)-\Delta^{-3}f(\zeta)-(1+3)\Delta^{-2}f(\zeta)-\frac{(1+3)^{(2)}}{2}\Delta^{-1}f(\zeta),$$
$$\frac{(2)^{(2)}}{2}f(\zeta+1)+\frac{(3)^{(2)}}{2}f(\zeta)=\Delta^{-3}f(\zeta+4)-\Delta^{-3}f(\zeta)-(4)\Delta^{-2}f(\zeta)-\frac{(4)^{(2)}}{2}\Delta^{-1}f(\zeta)$$
$$f(\zeta+1)+(3)f(\zeta)=\Delta^{-3}f(\zeta+4)-\Delta^{-3}f(\zeta)-(4)\Delta^{-2}f(\zeta)-\frac{(4)^{(2)}}{2}\Delta^{-1}f(\zeta) \tag{46}$$

Put $m=0$ in equation (112)

$$\frac{(2)^{(2)}}{2}f(\zeta)=\Delta^{-3}f(\zeta+3)-\Delta^{-3}f(\zeta)-(3)\Delta^{-2}f(\zeta)-\frac{(3)^{(2)}}{2}\Delta^{-1}f(\zeta)$$
$$f(\zeta)=\Delta^{-3}f(\zeta+3)-\Delta^{-3}f(\zeta)-(3)\Delta^{-2}f(\zeta)-\frac{(3)^{(2)}}{2}\Delta^{-1}f(\zeta) \tag{47}$$

$$0 = \Delta^{-3} f(\zeta+2) - \Delta^{-3} f(\zeta) - (2)\Delta^{-2} f(\zeta) - \frac{(2)^{(2)}}{2}\Delta^{-1} f(\zeta) \tag{48}$$

$$0 = \Delta^{-3} f(\zeta+1) - \Delta^{-3} f(\zeta) - (1)\Delta^{-2} f(\zeta) - \frac{(1)^{(2)}}{2}\Delta^{-1} f(\zeta) \tag{49}$$

$$0 = \Delta^{-3} f(\zeta) - \Delta^{-3} f(\zeta) \tag{50}$$

Adding equations (43) to (50), we get

$$f(\zeta+4) + (1+3) f(\zeta+3) + (1+3+6) f(\zeta+2) + (1+3+6+10) f(\zeta+1) + (1+3+6+10+15) f(\zeta) =$$

$$\Delta^{-4} f(\zeta+m+4) - \Delta^{-4} f(\zeta) - (m+4)\Delta^{-3} f(\zeta) - \frac{(m+4)^{(2)}}{2}\Delta^{-2} f(\zeta) - (1+3+6+10+15+21)\Delta^{-1} f(\zeta)$$

In general,

$$f(\zeta+4) + (4) f(\zeta+3) + (10) f(\zeta+2) + (20) f(\zeta+1) + (35) f(\zeta) = \Delta^{-4} f(\zeta+m+4) - \Delta^{-4} f(\zeta) - (m+4)\Delta^{-3} f(\zeta) - \frac{(m+4)^{(2)}}{2}\Delta^{-2} f(\zeta) - (56)\Delta^{-1} f(\zeta)$$

From the above result, we get the equation (113).

Example 2.19 Applying $f(\zeta) = \zeta^{(3)}$, $\zeta = 2$ and m=3 in equation (113), then we have

$$\text{LHS} = \sum_{r=0}^{3} \frac{(r+3)^{(3)}}{3^{(2)}} f(\zeta+3-r)$$

$$= \frac{(3)^{(3)}}{3^{(2)}}(2+3)^{(3)} + \frac{(4)^{(3)}}{3^{(2)}}(2+2)^{(3)} + \frac{(5)^{(3)}}{3^{(2)}}(2+1)^{(3)} + \frac{(6)^{(3)}}{3^{(2)}}(2)^{(3)}$$

$$= 5 + 4(4)^{(3)} + 10(3)^{(3)} + 20(2)^{(3)} = 60 + 4(24) + 10(6) + 20(0) = 216.$$

We know that, $\Delta^{-1}\zeta^{(3)} = \frac{\zeta^4}{4}$ $\Delta^{-2}\zeta^{(3)} = \frac{\zeta^5}{5.4}$,$\Delta^{-3}\zeta^{(3)} = \frac{\zeta^6}{6.5.4}$ and $\Delta^{-4}\zeta^{(3)} = \frac{\zeta^7}{7.6.5.4}$

$$\text{RHS} = \frac{(2+3+4)^{(7)}}{7^{(4)}} - \frac{(2)^{(7)}}{7^{(4)}} - 7\frac{(2)^{(6)}}{6^{(3)}} - \frac{(7)^{(2)}}{2}\frac{(2)^{(5)}}{3^{(2)}} + \frac{7^{(3)}}{3^{(2)}}\frac{2^{(4)}}{4}$$

$$= \frac{(9)^{(7)}}{7^{(4)}} - 0 - 0 - 0 - 0 = \frac{9.8.7.6.5.4.3}{7.6.5.4} = 216.$$

Therefore, LHS=RHS=216.

Theorem 2.20 If f be a real valued function and $\Delta^{-r} f(\zeta)$ where r=1,2,$\cdots$n exists, then

$$\sum_{r=0}^{m}\frac{(r+n-1)^{(n-1)}}{(n-1)!}f(\zeta+m-r)=\Delta^{-n}f(\zeta+m+n)-\Delta^{-n}f(\zeta)-\frac{(m+n)^{(1)}}{1!}\Delta^{-(n-1)}f(\zeta) -\frac{(m+n)^{(2)}}{2!}\Delta^{-(n-2)}f(\zeta) -\frac{(m+n)^{(n-1)}}{(n-1)!}\Delta^{-1}f(\zeta),\forall m\in Z^{+}. \quad (51)$$

Proof: The proof follows from, put $n=1$ in equation (114),

$$\sum_{r=0}^{m}f(\zeta+m-r)=\Delta^{-1}f(\zeta+m+1)-\Delta^{-1}f(\zeta)$$

Put $n=2$ in equation (114),

$$\sum_{r=0}^{m}(r+1)f(\zeta+m-r)=\Delta^{-2}f(\zeta+m+2)-\Delta^{-2}f(\zeta)-(m+2)\Delta^{-1}f(\zeta)$$

Put $n=3$ in equation (114),

$$\sum_{r=0}^{m}\frac{(r+2)^{(2)}}{2!}f(\zeta+m-r)=\Delta^{-3}f(\zeta+m+3)-\Delta^{-3}f(\zeta)-(m+3)\Delta^{-2}f(\zeta) -\frac{(m+3)^{(2!)}}{2}\Delta^{-1}f(\zeta)$$

Put $n=4$ in equation (114),

$$\sum_{r=0}^{m}\frac{(r+3)^{(3)}}{3!}f(\zeta+m-r)=\Delta^{-4}f(\zeta+m+4)-\Delta^{-4}f(\zeta)-(m+4)\Delta^{-3}f(\zeta) -\frac{(m+4)^{(2)}}{2!}\Delta^{-2}f(\zeta)-\frac{(m+4)^{(3)}}{3!}\Delta^{-1}f(\zeta).$$

Example 2.21 Put $n=5$ in equation (114)

$$\sum_{r=0}^{m}\frac{(r+4)^{(4)}}{4!}f(\zeta+m-r)=\Delta^{-5}f(\zeta+m+5)-\Delta^{-5}f(\zeta)-\frac{(m+5)^{(1)}}{1!}\Delta^{-4}f(\zeta) -\frac{(m+5)^{(2)}}{2!}\Delta^{-3}f(\zeta)-\frac{(m+5)^{(3)}}{3!}\Delta^{-2}f(\zeta)- \frac{(m+5)^{(4)}}{4!}\Delta^{-1}f(\zeta)$$

Applying $f(\zeta)=\zeta^{(3)}$, $\zeta=2$ and $m=3$ in the above equation, we get.

LHS$=\sum_{r=0}^{3}\frac{(r+4)^{(4)}}{4!}f(\zeta+3-r)$

$$=\frac{(4)^{(4)}}{4!}(2+3)^{(3)}+\frac{(5)^{(4)}}{4!}(2+2)^{(3)}+\frac{(6)^{(4)}}{4!}(2+1)^{(3)}+\frac{(7)^{(4)}}{4!}(2)^{(3)}$$
$$=5^{(3)}+5(4)^{(3)}+15(3)^{(3)}+35(2)^{(3)}=60+5(24)+15(6)+35(0)=270$$

We know that, $\Delta^{-1}\zeta^{(3)} = \frac{\zeta^4}{4}$ $\Delta^{-2}\zeta^{(3)} = \frac{\zeta^5}{5.4}$,$\Delta^{-3}\zeta^{(3)} = \frac{\zeta^6}{6.5.4}$, $\Delta^{-4}\zeta^{(3)} = \frac{\zeta^7}{7.6.5.4}$ and $\Delta^{-5}\zeta^{(3)} = \frac{\zeta^7}{8.7.6.5.4}$

RHS$= \frac{(2+3+3)^{(8)}}{8.7.6.5.4.3} - \frac{(2)^{(7)}}{7.6.5.4.3} - 8\frac{(2)^{(6)}}{6.5.4.3} - \frac{(8)^{(2)}}{2!}\frac{(2)^{(5)}}{60} - \frac{(8)^{(3)}}{3!}\frac{(2)^{(4)}}{48} - \frac{(8)^{(4)}}{4!}\frac{(2)^{(3)}}{3}$

$= \frac{(10.9.8.7.6.5.4)}{7.6.5.4.3} - 0 - 0 - 0 = 270$

Therefore, LHS=RHS=240.

Thus we have obtained Delta operator its inverse and some basic results using these operator.

3 $\alpha(q)$-Delta operator

In this section, we explore definition of $\alpha(q)$-Delta operator and also we derive generalized theorems of these operator which gives summation formulae related to these operator.

Definition 3.1 Let $f(t)$ be a real valued function and t, q, $\alpha \neq 0$ be a fixed real numbers. Then the $\alpha(q)$-Delta operator $\Delta_{\alpha(q)}$ is defined as

$$\Delta_{\alpha(q)} f(t) = f(tq) - \alpha f(t) \tag{52}$$

and the inverse of the $\alpha(q)$-Delta operator $\Delta^{-1}_{\alpha(q)}$ is defined as

$$f(t) = \Delta_{\alpha(q)} g(t) \implies \Delta^{-1}_{\alpha(q)} f(t) = g(t) + c$$

where c is an arbitrary constant.

Theorem 3.2 If f be a real valued function, $\alpha \neq 0$ and $\Delta^{-1}_{\alpha(q)} f(t)$ exists, then for every positive integer m,

$$f(tq^m) + \alpha f(tq^{m-1}) + \cdots + \alpha^m f(t) = \Delta^{-1}_{\alpha(q)} f(tq^{m+1}) - \alpha^{m+1}\Delta^{-1}_{\alpha(q)} f(t). \tag{53}$$

Proof: From equation (53), we have

$$\Delta_{\alpha(q)} f(t) = f(tq) - \alpha f(t)$$

Let $\Delta^{-1}_{\alpha(q)} f(t) = g(t)$

$$f(t) + \alpha g(t) = g(tq) \tag{54}$$

Replace t by tq in the above equation,

$$f(tq)+\alpha g(tq)=g(tq^2)$$

Substitute (54) in above equation,

$$f(tq)+\alpha f(t)+\alpha^2 g(t)=g(tq^2)$$

Again replace t by tq in the above equation,

$$f(tq^2)+\alpha f(tq)+\alpha^2 g(tq)=g(tq^3)$$

Substitute (54) in above equation,

$$f(tq^2)+\alpha f(tq)+\alpha^2 f(t)+\alpha^3 g(t)=g(tq^3)$$

In general, $f(tq^m)+\alpha f(tq^{m-1})+\cdots+\alpha^m f(t)+\alpha^{m+1}g(t)=g(tq^{m+1})$

$$\therefore f(tq^m)+\alpha f(tq^{m-1})+\cdots+\alpha^m f(t)=\Delta^{-1}_{\alpha(q)}f(tq^{m+1})-\alpha^{m+1}\Delta^{-1}_{\alpha(q)}f(t). \quad (55)$$

Example 3.3 Applying $f(t)=t$, $t=2$, $\alpha=1$, $m=2$, $q=3$ in equation (55), we get

$$\begin{aligned}\boldsymbol{LHS} &= f(tq^m)+\alpha f(tq^{m-1})+\alpha^2 f(tq^{m-2})+\cdots+\alpha^m f(t)\\ &=(tq^m)+\alpha(tq^{m-1})+\alpha^2(tq^{m-2})\\ &=(2)(3)^2+(1)(2)(3)^1+(1)^2(2)\\ &=26.\end{aligned}$$

We know that, $\Delta_{\alpha(q)}f(t)=f(tq)-\alpha f(t)$

$$\Delta_{\alpha(q)}(t)=(tq)-\alpha(t) \implies \Delta^{-1}_{\alpha(q)}t=\frac{t}{(q-\alpha)}$$

$$\begin{aligned}\boldsymbol{RHS} &= \Delta^{-1}_{\alpha(q)}f(tq^{m+1})-\alpha^{m+1}\Delta^{-1}_{\alpha(q)}f(t)\\ &=\left(\frac{tq^{m+1}}{q-\alpha}\right)-\alpha^{m+1}\left(\frac{t}{(q-\alpha)}\right)\\ &=\left(\frac{(2)(3)^3}{3-1}\right)-\alpha^3\left(\frac{2}{(3-1)}\right)\\ &=\left(\frac{(2)(27)}{2}\right)-(1)^3\left(\frac{2}{(2)}\right)\\ &=26\end{aligned}$$

$$\therefore LHS=RHS$$

Theorem 3.4 If f be a real valued function and $\alpha\neq 0$, $\Delta^{-r}_{\alpha(q)}f(t)$ where $r=1,2$ exists, then for every positive integer m,

$$f(tq^m)+2\alpha f(tq^{m-1})+3\alpha^2 f(tq^{m-2})+\cdots+(m+1)\alpha^m f(t)$$
$$=\Delta^{-2}_{\alpha(q)}f(tq^{m+2})-\alpha^{m+2}\Delta^{-2}_{\alpha(q)}f(t)-(m+2)\alpha^{m+1}\Delta^{-1}_{\alpha(q)}f(t).$$

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Proof: Consider,

$$f(tq^m)+\alpha f(tq^{m-1})+\cdots+\alpha^m f(t)=\Delta^{-1}_{\alpha(q)}f(tq^{m+1})-\alpha^{m+1}\Delta^{-1}_{\alpha(q)}f(t) \tag{56}$$

Applying $\Delta^{-1}_{\alpha(q)}$ in the equation (53),
$\Delta^{-1}_{\alpha(q)}f(tq^m)+\alpha\Delta^{-1}_{\alpha(q)}f(tq^{m-1})+\alpha^2\Delta^{-1}_{\alpha(q)}f(tq^{m-2})+\cdots+\alpha^m\Delta^{-1}_{\alpha(q)}f(t)$

$$=\Delta^{-2}_{\alpha(q)}f(tq^{m+1})-\alpha^{m+1}\Delta^{-2}_{\alpha(q)}f(t) \tag{57}$$

Suppose $m=4,3,2,1$ and 0 in the equation (57), we get

$$f(tq^4)+\alpha f(tq^3)+\alpha^2 f(tq^2)+\alpha^3 f(tq)+\alpha^4 f(t)=\Delta^{-1}_{\alpha(q)}f(tq^5)-\alpha^5\Delta^{-1}_{\alpha(q)}f(t) \tag{58}$$

$$f(tq^3)+\alpha f(tq^2)+\alpha^2 f(tq)+\alpha^3 f(t)=\Delta^{-1}_{\alpha(q)}f(tq^4)-\alpha^4\Delta^{-1}_{\alpha(q)}f(t) \tag{59}$$

$$f(tq^2)+\alpha f(tq)+\alpha^2 f(t)=\Delta^{-1}_{\alpha(q)}f(tq^3)-\alpha^3\Delta^{-1}_{\alpha(q)}f(t) \tag{60}$$

$$f(tq^1)+\alpha f(t)=\Delta^{-1}_{\alpha(q)}f(tq^2)-\alpha^2\Delta^{-1}_{\alpha(q)}f(t) \tag{61}$$

$$f(t)=\Delta^{-1}_{\alpha(q)}f(tq)-\alpha\Delta^{-1}_{\alpha(q)}f(t) \tag{62}$$

$$0=\Delta^{-1}_{\alpha(q)}f(t)-\Delta^{-1}_{\alpha(q)}f(t) \tag{63}$$

Multiply by α, α^2, α^3, α^4, α^5 in the equation (59), (60), (61), (62) and (63) respectively,

$$\alpha f(tq^3)+\alpha^2 f(tq^2)+\alpha^3 f(tq)+\alpha^4 f(t)=\alpha\Delta^{-1}_{\alpha(q)}f(tq^4)-\alpha^5\Delta^{-1}_{\alpha(q)}f(t) \tag{64}$$

$$\alpha^2 f(tq^2)+\alpha^3 f(tq)+\alpha^4 f(t)=\alpha^2\Delta^{-1}_{\alpha(q)}f(tq^3)-\alpha^5\Delta^{-1}_{\alpha(q)}f(t) \tag{65}$$

$$\alpha^3 f(tq^1)+\alpha^4 f(t)=\alpha^3\Delta^{-1}_{\alpha(q)}f(tq^2)-\alpha^5\Delta^{-1}_{\alpha(q)}f(t) \tag{66}$$

$$\alpha^4 f(t)=\alpha^4\Delta^{-1}_{\alpha(q)}f(tq)-\alpha^5\Delta^{-1}_{\alpha(q)}f(t) \tag{67}$$

$$\alpha^5(0)=\alpha^5\Delta^{-1}_{\alpha(q)}f(t)-\alpha^5\Delta^{-1}_{\alpha(q)}f(t) \tag{68}$$

Adding the equation (58), (64) to (68), we get
$f(tq^4)+2\alpha f(tq^3)+3\alpha^2 f(tq^2)+4\alpha^3 f(tq)+5\alpha^4 f(t)$
$$=\Delta^{-2}_{\alpha(q)}f(tq^6)-\alpha^6\Delta^{-2}_{\alpha(q)}f(t)-6\alpha^5\Delta^{-1}_{\alpha(q)}f(t)$$
In general,

$$f(tq^m)+2\alpha f(tq^{m-1})+3\alpha^2 f(tq^{m-2})+\cdots+(m+1)\alpha^m f(t)$$
$$=\Delta^{-2}_{\alpha(q)}f(tq^{m+2})-\alpha^{m+2}\Delta^{-2}_{\alpha(q)}f(t)-(m+2)\alpha^{m+1}\Delta^{-1}_{\alpha(q)}f(t). \qquad (69)$$

Example 3.5 Applying $f(t)=t$, $t=3$, $q=4$, $\alpha=1$ & $m=2$ in equation (69),
LHS$=f(tq^m)+2\alpha f(tq^{m-1})+3\alpha^2 f(tq^{m-2})+\cdots+(m+1)\alpha^m f(t)$
$=(tq^m)+2\alpha(tq^{m-1})+3\alpha^2(tq^{m-2})$
$=(3)(4^2)+2(1)((3)(4))+(3)(1)((3))$
We know that, $\Delta_{\alpha(q)}f(t)=f(tq)-\alpha f(t)$
$\implies \Delta^{-1}_{\alpha(q)}t=\left(\frac{t}{q-\alpha}\right)$, $\Delta^{-2}_{\alpha(q)}t=\left(\frac{t}{(q-\alpha)^2}\right)$
RHS $=\Delta^{-2}_{\alpha(q)}f(tq^{m+2})-\alpha^{m+2}\Delta^{-2}_{\alpha(q)}f(t)-(m+2)\alpha^{m+1}\Delta^{-1}_{\alpha(q)}f(t)$
$=\left(\frac{tq^{m+2}}{(q-\alpha)^2}\right)-\alpha^{m+2}\left(\frac{t}{(q-\alpha)^2}\right)-(m+2)\alpha^{m+1}\left(\frac{t}{(q-\alpha)}\right)$
$=\left(\frac{(3)(4)^4}{(3^2)}\right)-(1)^4\left(\frac{3}{3^2}\right)-(4)(1)^3\left(\frac{3}{3}\right)$
$=81$

$$\therefore LHS=RHS$$

Theorem 3.6 If f be a real valued function, $\alpha\neq 0$ and $\Delta^{-r}_{\alpha(q)}f(t)$ where $r=1,2,3$ exists, then for every positive integer m,

$$\sum_{r=0}^{m}\alpha^r\left(\frac{(r+2)^{(2)}}{2}\right)f(tq^{m-r})=\Delta^{-3}_{\alpha(q)}f(tq^{m+3})-\alpha^{m+3}\Delta^{-3}_{\alpha(q)}f(t)-(m+3)\alpha^{m+2}\Delta^{-2}_{\alpha(q)}f(t)-\left(\frac{(m+3)^{(2)}}{2}\right)\alpha^{m+1}\Delta^{-1}_{\alpha(q)}f(t).$$

Proof: Consider,

$$f(tq^m)+2\alpha f(tq^{m-1})+3\alpha^2 f(tq^{m-2})+\cdots+(m+1)\alpha^m f(t)$$
$$=\Delta^{-2}_{\alpha(q)}f(tq^{m+2})-\alpha^{m+2}\Delta^{-2}_{\alpha(q)}f(t)-(m+2)\alpha^{m+1}\Delta^{-1}_{\alpha(q)}f(t) \qquad (70)$$

Applying $\Delta^{-1}_{\alpha(q)}$ in the equation (53),
$\Delta^{-2}_{\alpha(q)}f(tq^m)+\alpha\Delta^{-2}_{\alpha(q)}f(tq^{m-1})+\cdots+\alpha^m\Delta^{-2}_{\alpha(q)}f(t)=\Delta^{-3}_{\alpha(q)}f(tq^{m+1})-\alpha^{m+1}\Delta^{-3}_{\alpha(q)}f(t)$

Suppose $m = 4, 3, 2, 1$ and 0 in (70), we get

$$f(tq^4)+2\alpha f(tq^3)+3\alpha^2 f(tq^2)+4\alpha^3 f(t)+5\alpha^4 f(t)=\Delta^{-2}_{\alpha(q)}f(tq^6)-\alpha^6\Delta^{-2}_{\alpha(q)}f(t)-(6)\alpha^5\Delta^{-1}_{\alpha(q)}f(t) \tag{71}$$

$$f(tq^3)+2\alpha f(tq^2)+3\alpha^2 f(tq)+4\alpha^3 f(t)=\Delta^{-2}_{\alpha(q)}f(tq^5)-\alpha^5\Delta^{-2}_{\alpha(q)}f(t)-(5)\alpha^4\Delta^{-1}_{\alpha(q)}f(t) \tag{72}$$

$$f(tq^2)+2\alpha f(tq)+3\alpha^2 f(t)=\Delta^{-2}_{\alpha(q)}f(tq^4)-\alpha^4\Delta^{-2}_{\alpha(q)}f(t)-(4)\alpha^3\Delta^{-1}_{\alpha(q)}f(t) \tag{73}$$

$$f(tq)+2\alpha f(t)=\Delta^{-2}_{\alpha(q)}f(tq^3)-\alpha^3\Delta^{-2}_{\alpha(q)}f(t)-(3)\alpha^2\Delta^{-1}_{\alpha(q)}f(t) \tag{74}$$

$$f(t)=\Delta^{-2}_{\alpha(q)}f(tq^2)-\alpha^2\Delta^{-2}_{\alpha(q)}f(t)-(2)\alpha\Delta^{-1}_{\alpha(q)}f(t) \tag{75}$$

$$0=\Delta^{-2}_{\alpha(q)}f(tq)-\alpha\Delta^{-2}_{\alpha(q)}f(t)-\Delta^{-1}_{\alpha(q)}f(t) \tag{76}$$

$$0=\Delta^{-2}_{\alpha(q)}f(t)-\Delta^{-2}_{\alpha(q)}f(t) \tag{77}$$

Multiply by α, α^2, α^3, α^4, α^5, α^6 in equation (72), (73), (74), (75), (76) & (77),

$$\alpha f(tq^3)+2\alpha^2 f(tq^2)+3\alpha^3 f(tq)+4\alpha^4 f(t)=\alpha\Delta^{-2}_{\alpha(q)}f(tq^5)-\alpha^6\Delta^{-2}_{\alpha(q)}f(t)-(5)\alpha^5\Delta^{-1}_{\alpha(q)}f(t) \tag{78}$$

$$\alpha^2 f(tq^2)+2\alpha^3 f(tq)+3\alpha^4 f(t)=\alpha^2\Delta^{-2}_{\alpha(q)}f(tq^4)-\alpha^6\Delta^{-2}_{\alpha(q)}f(t)-(4)\alpha^5\Delta^{-1}_{\alpha(q)}f(t) \tag{79}$$

$$\alpha^3 f(tq)+2\alpha^4 f(t)=\alpha^3\Delta^{-2}_{\alpha(q)}f(tq^3)-\alpha^6\Delta^{-2}_{\alpha(q)}f(t)-(3)\alpha^5\Delta^{-1}_{\alpha(q)}f(t) \tag{80}$$

$$\alpha^4 f(t)=\alpha^4\Delta^{-2}_{\alpha(q)}f(tq^2)-\alpha^6\Delta^{-2}_{\alpha(q)}f(t)-(2)\alpha^5\Delta^{-1}_{\alpha(q)}f(t) \tag{81}$$

$$\alpha^5(0)=\alpha^5\Delta^{-2}_{\alpha(q)}f(tq)-\alpha^6\Delta^{-2}_{\alpha(q)}f(t)-\alpha^5\Delta^{-1}_{\alpha(q)}f(t) \tag{82}$$

$$\alpha^6(0)=\alpha^6\Delta^{-2}_{\alpha(q)}f(t)-\alpha^6\Delta^{-2}_{\alpha(q)}f(t) \tag{83}$$

Adding equation (71), (78) to (83), we get

$$f(tq^4)+\{\alpha+2\alpha\}f(tq^3)+\{\alpha^2+2\alpha^2+3\alpha^2\}f(tq^2)+\{\alpha^3+2\alpha^3+3\alpha^3+4\alpha^3\}f(tq^1)$$
$$+\{\alpha^4+2\alpha^4+3\alpha^4+4\alpha^4+5\alpha^4\}f(t)$$
$$=\Delta^{-3}_{\alpha(q)}f(tq^7)-\alpha^7\Delta^{-3}_{\alpha(q)}f(t)-7\alpha^6\Delta^{-2}_{\alpha(q)}f(t)-\left(\frac{7^{(2)}}{2}\right)\alpha^5\Delta^{-1}_{\alpha(q)}f(t)$$

LHS$=\left(\frac{2^{(2)}}{2}\right)f(tq^4)+\alpha\left(\frac{3^{(2)}}{2}\right)f(tq^3)+\alpha^2\left(\frac{4^{(2)}}{2}\right)f(tq^2)+\alpha^3\left(\frac{5^{(2)}}{2}\right)f(tq^1)$

$$+\alpha^4\left(\frac{6^{(2)}}{2}\right)f(t)$$

$$= \sum_{r=0}^{4} \alpha^r \left(\frac{(r+2)^{(2)}}{2} \right) f(tq^{4-r})$$

$$\therefore \sum_{r=0}^{4} \alpha^r \left(\frac{(r+2)^{(2)}}{2} \right) f(tq^{4-r}) = \Delta^{-3}_{\alpha(q)} f(tq^7) - \alpha^7 \Delta^{-3}_{\alpha(q)} f(t) - 7\alpha^6 \Delta^{-2}_{\alpha(q)} f(t) - \left(\frac{7^{(2)}}{2} \right) \alpha^5 \Delta^{-1}_{\alpha(q)} f(t)$$

In general,

$$\therefore \sum_{r=0}^{m} \alpha^r \left(\frac{(r+2)^{(2)}}{2} \right) f(tq^{m-r})$$

$$= \Delta^{-3}_{\alpha(q)} f(tq^{m+3}) - \alpha^{m+3} \Delta^{-3}_{\alpha(q)} f(t) - (m+3)\alpha^{m+2} \Delta^{-2}_{\alpha(q)} f(t) - \left(\frac{(m+3)^{(2)}}{2} \right) \alpha^{m+1} \Delta^{-1}_{\alpha(q)} f(t). \tag{84}$$

Example 3.7 Applying $f(t) = t$, $t = 2$, $\alpha = 3$, $m = 3$, $q = 6$, in the equation (84), we get

$$\boldsymbol{LHS} = \sum_{r=0}^{m} \alpha^r \left(\frac{(r+2)^{(2)}}{2} \right) f(tq^{m-r})$$

$$= \sum_{r=0}^{3} 3^r \left(\frac{(r+2)^{(2)}}{2} \right) (tq^{m-r})$$

$$= \left\{ (3)^0 \left(\frac{2^{(2)}}{2} \right) (2)(6^{3-0}) + (3)^1 \left(\frac{3^{(2)}}{2} \right) (2)(6^{3-1}) + (3)^2 \left(\frac{4^{(2)}}{2} \right) (2)(6^{3-2}) + (3)^3 \left(\frac{5^{(2)}}{2} \right) (2)(6^{3-3}) \right\}$$

$$= 2268$$

We know that,

$$\Delta^{-1}_{\alpha(q)} t = \frac{t}{(q-\alpha)},\ \Delta^{-2}_{\alpha(q)} t = \frac{t}{(q-\alpha)^2},\ \Delta^{-3}_{\alpha(q)} t = \frac{t}{(q-\alpha)^3}$$

$$= \Delta^{-3}_{\alpha(q)} f(tq^{m+3}) - \alpha^{m+3} \Delta^{-3}_{\alpha(q)} f(t) - (m+3)\alpha^{m+2} \Delta^{-2}_{\alpha(q)} f(t) - \left(\frac{(m+3)^{(2)}}{2} \right) \alpha^{m+1} \Delta^{-1}_{\alpha(q)} f(t)$$

$$= \frac{(2)(6^6)}{(3^3)} - (3^6)\frac{(2)}{(6-3)^3} - (6)(3^5)\left(\frac{(2)}{(6-3)^2} \right) - \frac{(6)^{(2)}}{2}(3^4)\left(\frac{(2)}{(6-3)} \right)$$

$$= 2268.$$

$$LHS = RHS$$

Theorem 3.8 If f be a real valued function, $\alpha \neq 0$ and $\Delta^{-r}_{\alpha(q)} f(t)$ where $r = 1, 2, 3, 4$ exists,then for every positive integer m,

$$\sum_{r=0}^{m} \alpha^r \left(\frac{(r+3)^{(3)}}{3^{(2)}} \right) f(tq^{m-r}) = \Delta^{-4}_{\alpha(q)} f(tq^{m+4}) - \alpha^{m+4} \Delta^{-4}_{\alpha(q)} f(t) - (m+4)\alpha^{m+3} \Delta^{-3}_{\alpha(q)} f(t) - \left(\frac{(m+4)^{(2)}}{2} \right) \alpha^{m+2} \Delta^{-2}_{\alpha(q)} f(t) - \left(\frac{(m+4)^{(3)}}{3^{(2)}} \right) \alpha^{m+1} \Delta^{-1}_{\alpha(q)} f(t). \tag{85}$$

Example 3.9 Applying $f(t) = t$, $t = 3$, $q = 5$, $m = 2$ & $\alpha = 3$ in the equation (85), we get

$$\begin{aligned} \boldsymbol{LHS} &= \sum_{r=0}^{m} \alpha^r \left(\frac{(r+3)^{(3)}}{3^{(2)}} \right) f(tq^{m-r}) \\ &= \sum_{r=0}^{2} (3)^r \left(\frac{(r+3)^{(3)}}{3^{(2)}} \right) (tq^{2-r}) \\ &= 3^{(0)} \left(\frac{3^{(3)}}{3^{(2)}} \right) (3)(5^2) + 3^{(1)} \left(\frac{4^{(3)}}{3^{(2)}} \right) (3)(5^1) + 3^{(2)} \left(\frac{5^{(3)}}{3^{(2)}} \right) (3) = 525 \end{aligned}$$

We know that,

$$\Delta^{-1}_{\alpha(q)} t = \frac{t}{(q-\alpha)}, \ \Delta^{-2}_{\alpha(q)} t = \frac{t}{(q-\alpha)^2},$$

$$\Delta^{-3}_{\alpha(q)} t = \frac{t}{(q-\alpha)^3}, \ \Delta^{-4}_{\alpha(q)} t = \frac{t}{(q-\alpha)^4}$$

$$\begin{aligned} \boldsymbol{RHS} &= \Big\{ \Delta^{-4}_{\alpha(q)} f(tq^{m+4}) - \alpha^{m+4} \Delta^{-4}_{\alpha(q)} f(t) - (m+4)\alpha^{m+3} \Delta^{-3}_{\alpha(q)} f(t) \\ &\qquad - \left(\frac{(m+4)^{(2)}}{2} \right) \alpha^{m+2} \Delta^{-2}_{\alpha(q)} f(t) - \left(\frac{(m+4)^{(3)}}{3^{(2)}} \right) \alpha^{m+1} \Delta^{-1}_{\alpha(q)} f(t) \Big\} \\ &= \Big\{ \left(\frac{(3)(5^6)}{(5-3)^4} \right) - (3)^6 \left(\frac{(3)}{(5-3)^4} \right) - (6)(3^5) \left(\frac{(3)}{(5-3)^3} \right) \\ &\qquad - \left(\frac{(6)(5)}{2} \right) (3^4) \left(\frac{(3)}{(5-3)^2} \right) - \left(\frac{(6)(5)(4)}{(3)(2)} \right) (3^3) \left(\frac{(3)}{(5-3)^1} \right) \Big\} \\ &= 525 \end{aligned}$$

$$\therefore LHS = RHS$$

Theorem 3.10 If f be a real valued function, $\alpha \neq 0$ and $\Delta^{-r}_{\alpha(q)} f(t)$ where $r = 1, 2, \cdots, n$ exists, then for every positive integer m,

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$$\sum_{r=0}^{m}\alpha^r\left(\frac{(r+n-1)^{(n-1)}}{(n-1)!}\right)f(tq^{m-r})$$
$$=\left\{\Delta_{\alpha(q)}^{-n}f(tq^{m+n})-\alpha^{m+n}\Delta_{\alpha(q)}^{-n}f(t)-\left(\frac{(m+n)^{(1)}}{1!}\right)\alpha^{m+(n-1)}\Delta_{\alpha(q)}^{-(n-1)}f(t)\right.$$
$$\left.-\left(\frac{(m+n)^{(2)}}{2!}\right)\alpha^{m+(n-2)}\Delta_{\alpha(q)}^{-(n-2)}f(t)-\cdots-\left(\frac{(m+n)^{(n-1)}}{(n-1)!}\right)\alpha^{m+1}\Delta_{\alpha(q)}^{-1}f(t)\right\}. \tag{86}$$

Proof: For $n=1$ in the equation (86), which gives

$$f(tq^m)+\alpha f(tq^{m-1})+\alpha^2 f(tq^{m-2})+\cdots+\alpha^m f(t)=\Delta_{\alpha(q)}^{-1}f(tq^{m+1})-\alpha^{m+1}\Delta_{\alpha(q)}^{-1}f(t)$$

For $n=2$ in the equation (86), which gives

$$f(tq^m)+2\alpha f(tq^{m-1})+3\alpha^2 f(tq^{m-2})+\cdots+(m+1)\alpha^m f(t)$$
$$=\Delta_{\alpha(q)}^{-2}f(tq^{m+2})-\alpha^{m+2}\Delta_{\alpha(q)}^{-2}f(t)-(m+2)\alpha^{m+1}\Delta_{\alpha(q)}^{-1}f(t)$$

. For $n=3$ in the equation (86), which gives

$$\sum_{r=0}^{m}\alpha^r\left(\frac{(r+2)^{(2)}}{2}\right)f(tq^{m-r}) =$$
$$\Delta_{\alpha(q)}^{-3}f(tq^{m+3})-\alpha^{m+3}\Delta_{\alpha(q)}^{-3}f(t)-(m+3)\alpha^{m+2}\Delta_{\alpha(q)}^{-2}f(t)$$
$$-\left(\frac{(m+3)^{(2)}}{2}\right)\alpha^{m+1}\Delta_{\alpha(q)}^{-1}f(t)$$

For $n=4$ in the equation (86), which gives

$$\sum_{r=0}^{m}\alpha^r\left(\frac{(r+3)^{(3)}}{3^{(2)}}\right)f(tq^{m-r})$$
$$=\Delta_{\alpha(q)}^{-4}f(tq^{m+4})-\alpha^{m+4}\Delta_{\alpha(q)}^{-4}f(t)-(m+4)\alpha^{m+3}\Delta_{\alpha(q)}^{-3}f(t)$$
$$-\left(\frac{(m+4)^{(2)}}{2}\right)\alpha^{m+2}\Delta_{\alpha(q)}^{-2}f(t)-\left(\frac{(m+4)^{(3)}}{3^{(2)}}\right)\alpha^{m+1}\Delta_{\alpha(q)}^{-1}f(t)$$

Proceeding this upto n times, we have

$$\therefore\sum_{r=0}^{m}\alpha^r\left(\frac{(r+n-1)^{(n-1)}}{(n-1)!}\right)f(tq^{m-r})$$
$$=\left\{\Delta_{\alpha(q)}^{-n}f(tq^{m+n})-\alpha^{m+n}\Delta_{\alpha(q)}^{-n}f(t)-\left(\frac{(m+n)^{(1)}}{1!}\right)\alpha^{m+(n-1)}\Delta_{\alpha(q)}^{-(n-1)}f(t)\right.$$
$$\left.-\left(\frac{(m+n)^{(2)}}{2!}\right)\alpha^{m+(n-2)}\Delta_{\alpha(q)}^{-(n-2)}f(t)-\cdots-\left(\frac{(m+n)^{(n-1)}}{(n-1)!}\right)\alpha^{m+1}\Delta_{\alpha(q)}^{-1}f(t)\right\}.$$

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Thus, we have derived higher order theorems of the $\alpha(q)$-Delta operator which gives the summation formulae related to these operators and these results are validated with suitable examples.

4 q-Nabla operator

In this section, we explore the basic definition of q-Nabla operator and summation formulae involving inverse of higher order q-Nabla operator.

Definition 4.1 Let $f(t)$ be a real valued function and t , q be a fixed real number. Then, q-Nabla operator ∇_q is defined as

$$\nabla_q f(t) = f(t) - f(t/q) \tag{87}$$

and the inverse of the q-Nabla operator ∇_q^{-1} is defined as

$$f(t) = \nabla_q g(t) \implies \nabla_q^{-1} f(t) = g(t) + c$$

where c is an arbitrary constant.

Theorem 4.2 If f be a real valued function and $\nabla_q^{-1} f(t)$ exists, then for every positive integer m,

$$f(tq^m) + f(tq^{m-1}) + f(tq^{m-2}) + \cdots + f(t) = \nabla_q^{-1} f(tq^{m+1}) - \nabla_q^{-1} f(t). \tag{88}$$

Proof: From equation (88),we have

$$\nabla_q f(t) = f(t) - f(t/q)$$

Let $\nabla_q^{-1} f(t) = g(t)$

$$f(t) = \nabla_q g(t)$$

$$\implies f(t) = g(t) - g(t/q)$$

$$f(t) + g(t/q) = g(t) \tag{89}$$

Replace t by tq in the above equation,

$$f(tq) + g(t) = g(tq)$$

Substitute (89) in above equation,

$$f(tq) + f(t) + g(t/q) = g(tq)$$

Again replace t by tq in the above equation,

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$$f(tq^2)+f(tq)+g(t)=g(tq^2)$$

Substitute (89) in above equation,

$$f(tq^2)+f(tq)+f(t)+g(t/q)=g(tq^2)$$

In general, $f(tq^m)+f(tq^{m-1})+f(tq^{m-2})+\cdots+f(t)+g(t/q)=g(tq^m)$

$$\therefore f(tq^m)+f(tq^{m-1})+f(tq^{m-2})+\cdots+f(t)=\nabla_q^{-1}f(tq^m)-\nabla_q^{-1}f(t/q). \qquad (90)$$

Example 4.3 Applying $f(t)=t$, $t=3$, $q=2$, $m=2$ in the equation (90), we get

$\boldsymbol{LHS} = f(tq^m)+f(tq^{m-1})+f(tq^{m-2})+\cdots+f(t)$

$= (tq^m)+(tq^{m-1})+(tq^{m-2})$

$= (3)(2^2)+(3)(2^1)+(3)=21.$

We know that, $\nabla_q t = t - t/q = t\left(1-\frac{1}{q}\right) = t\left(\frac{q-1}{q}\right)$

$\therefore \nabla_q^{-1} t = t\left(\frac{q}{q-1}\right)$

$\boldsymbol{RHS} = \nabla_q^{-1}f(tq^m)-\nabla_q^{-1}f(t/q)$

$= (tq^m)\left(\frac{q}{q-1}\right)-\left(\frac{t}{q}\right)\left(\frac{q}{q-1}\right)$

$= (3)(2^2)\left(\frac{2}{1}\right)-\left(\frac{3}{2}\right)\left(\frac{2}{1}\right)$

$= 21.$

Theorem 4.4 If f be a real valued function $\nabla_q^{-r}f(t)$where $r=1,2$ exists, then for every positive integer m,
$f(tq^m)+2f(tq^{m-1})+\cdots+(m+1)f(t)=\left\{\nabla_q^{-2}f(tq^m)-\nabla_q^{-2}f(t/q)-(m+1)\nabla_q^{-1}f(t/q)\right\}.$

Proof: Consider,

$$f(tq^m)+f(tq^{m-1})+f(tq^{m-2})+\cdots+f(t)=\nabla_q^{-1}f(tq^m)-\nabla_q^{-1}f(t/q) \qquad (91)$$

Suppose $m=4,3,2,1$ and 0 in the equation (91), we get

$$f(tq^4)+f(tq^3)+f(tq^2)+f(tq)+f(t)=\nabla_q^{-1}f(tq^4)-\nabla_q^{-1}f(t/q) \qquad (92)$$

$$f(tq^3)+f(tq^2)+f(tq)+f(t)=\nabla_q^{-1}f(tq^3)-\nabla_q^{-1}f(t/q) \qquad (93)$$

$$f(tq^2)+f(tq)+f(t)=\nabla_q^{-1}f(tq^2)-\nabla_q^{-1}f(t/q) \qquad (94)$$

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$$f(tq)+f(t)=\nabla_q^{-1}f(tq)-\nabla_q^{-1}f(t/q) \tag{95}$$

$$f(t)=\nabla_q^{-1}f(t)-\nabla_q^{-1}f(t/q) \tag{96}$$

Adding the equations (92) to (96), we get
$f(tq^4)+2f(tq^3)+3f(tq^2)+4f(tq)+5f(t)=\nabla_q^{-2}f(tq^4)-\nabla_q^{-2}f(t/q)-5\nabla_q^{-1}f(t/q)$
In general,

$$\therefore f(tq^m)+2f(tq^{m-1})+\cdots+(m+1)f(t)=\nabla_q^{-2}f(tq^m)-\nabla_q^{-2}f(t/q)-(m+1)\nabla_q^{-1}f(t/q). \tag{97}$$

Example 4.5 Applying $f(t)=t$, $t=5$, $q=3$, $m=2$ in the equation (97), we get
$\boldsymbol{LHS} = f(tq^m)+2f(tq^{m-1})+3f(tq^{m-2})+\cdots+(m+1)f(t)$
$= (tq^m)+2(tq^{m-1})+3(tq^{m-2})$
$= (5)(3^2)+(2)((5)(3^1))+(3)((5)(3^0))$
$= 90.$
We know that, $\nabla_q^{-1}t=t\left(\frac{q}{q-1}\right)$, $\nabla_q^{-2}t=t\left(\frac{q}{q-1}\right)^2$
$\boldsymbol{RHS} = \nabla_q^{-2}f(tq^m)-\nabla_q^{-2}f(t/q)-(m+1)\nabla_q^{-1}f(t/q)$
$= \left(\frac{q}{q-1}\right)^2(tq^m)-\left(\frac{q}{q-1}\right)^2(t/q)-(m+1)\left(\frac{t}{q}\right)\left(\frac{q}{q-1}\right)$
$= \left(\frac{3}{2}\right)^2(5)(3^2)-\left(\frac{3}{2}\right)^2(5/3)-(3)\left(\frac{5}{3}\right)\left(\frac{3}{2}\right)$
$= 90.$

$$\therefore LHS = RHS$$

Theorem 4.6 If f be a real valued function $\nabla_q^{-r}f(t)$ where $r=1,2,3$ exists, then for every positive integer m,

$$\sum_{r=0}^{m}\left(\frac{(r+2)^{(2)}}{2}\right)f(tq^{m-r}) = \nabla_q^{-3}f(tq^m)-\nabla_q^{-3}f(t/q)-(m+1)\nabla_q^{-2}f(t/q)-\left(\frac{(m+2)^{(2)}}{2}\right)\nabla_q^{-1}f(t/q).$$

Proof: Consider,

$$f(tq^m)+2f(tq^{m-1})+\cdots+(m+1)f(t)=\nabla_q^{-2}f(tq^m)-\nabla_q^{-2}f(t/q)-(m+1)\nabla_q^{-1}f(t/q) \tag{98}$$

Suppose $m=4,3,2,1$ and 0 in the equation (98), we get

$$f(tq^4)+2f(tq^3)+3f(tq^2)+4f(tq)+5f(t)=\nabla_q^{-2}f(tq^4)-\nabla_q^{-2}f(t/q)-5\nabla_q^{-1}f(t/q) \tag{99}$$

$$f(tq^3)+2f(tq^2)+3f(tq)+4f(t)=\nabla_q^{-2}f(tq^3)-\nabla_q^{-2}f(t/q)-4\nabla_q^{-1}f(t/q) \tag{100}$$

$$f(tq^2)+2f(tq)+3f(t)=\nabla_q^{-2}f(tq^2)-\nabla_q^{-2}f(t/q)-3\nabla_q^{-1}f(t/q) \tag{101}$$

$$f(tq)+2f(t)=\nabla_q^{-2}f(tq)-\nabla_q^{-2}f(t/q)-2\nabla_q^{-1}f(t/q) \tag{102}$$

$$f(t)=\nabla_q^{-2}f(t)-\nabla_q^{-2}f(t/q)-\nabla_q^{-1}f(t/q) \tag{103}$$

Adding the equations (99) to (103), we get

$$f(tq^4)+(1+2)f(tq^3)+(1+2+3)f(tq^2)+(1+2+3+4)f(tq)+(1+2+3+4+5)f(t)$$
$$=\nabla_q^{-3}f(tq^4)-\nabla_q^{-3}f(t/q)-5\nabla_q^{-2}f(t/q)-(1+2+3+4+5)\nabla_q^{-1}f(t/q)$$

In general,

$$f(tq^m)+\left(\frac{3^{(2)}}{2}\right)f(tq^{m-1})+\left(\frac{4^{(2)}}{2}\right)f(tq^{m-2})+\left(\frac{5^{(2)}}{2}\right)f(tq^{m-3})+\left(\frac{6^{(2)}}{2}\right)f(t)$$
$$=\nabla_q^{-3}f(tq^m)-\nabla_q^{-3}f(t/q)-(m+1)\nabla_q^{-2}f(t/q)-\left(\frac{(6)^{(2)}}{2}\right)\nabla_q^{-1}f(t/q)$$

$$\therefore \sum_{r=0}^{m}\left(\frac{(r+2)^{(2)}}{2}\right)f(tq^{m-r})$$
$$=\nabla_q^{-3}f(tq^m)-\nabla_q^{-3}f(t/q)-(m+1)\nabla_q^{-2}f(t/q)-\left(\frac{(m+2)^{(2)}}{2}\right)\nabla_q^{-1}f(t/q). \tag{104}$$

Example 4.7 Applying $f(t)=t$, $t=3$, $q=6$, $m=2$ in the equation (104), we get

$$\boldsymbol{LHS}=\sum_{r=0}^{m}\left(\frac{(r+2)^{(2)}}{2}\right)f(tq^{m-r})$$
$$=\sum_{r=0}^{2}\left(\frac{(r+2)^{(2)}}{2}\right)(tq^{2-r})$$

$$= \left(\frac{2^{(2)}}{2}\right)(tq^2) + \left(\frac{3^{(2)}}{2}\right)(tq^1) + \left(\frac{4^{(2)}}{2}\right)(tq^0) = 180.$$

We know that, $\nabla_q^{-1} t = t\left(\frac{q}{q-1}\right)$, $\nabla_q^{-2} t = t\left(\frac{q}{q-1}\right)^2$, $\nabla_q^{-3} t = t\left(\frac{q}{q-1}\right)^3$

$$\boldsymbol{RHS} = \nabla_q^{-3} f(tq^m) - \nabla_q^{-3} f(t/q) - (m+1)\nabla_q^{-2} f(t/q) - \left(\frac{(m+2)^{(2)}}{2}\right)\nabla_q^{-1} f(t/q) =$$

$$(tq^m)\left(\frac{q}{q-1}\right)^3 - (t/q)\left(\frac{q}{q-1}\right)^3 - (3)(t/q)\left(\frac{q}{q-1}\right)^2 - \left(\frac{(4)(3)}{2}\right)(t/q)\left(\frac{q}{q-1}\right)$$

$$= (3)(6^2)\left(\frac{6}{5}\right)^3 - (3/6)\left(\frac{6}{5}\right)^3 - (3)(3/6)\left(\frac{6}{5}\right)^2 - \left(\frac{(4)(3)}{2}\right)(3/6)\left(\frac{6}{5}\right) = 180.$$

$$\therefore LHS = RHS$$

Theorem 4.8 If f be a real valued function $\nabla_q^{-r} f(t)$ where $r = 1,2,3,4$ exists, then for every positive integer m,

$$\sum_{r=0}^{m}\left(\frac{(r+3)^{(3)}}{3^{(2)}}\right) f(tq^{m-r})$$
$$= \nabla_q^{-4} f(tq^m) - \nabla_q^{-4} f(t/q) - (m+1)\nabla_q^{-3} f(t/q) - \left(\frac{(m+2)^{(2)}}{2}\right)\nabla_q^{-2} f(t/q)$$
$$- \left(\frac{(m+3)^{(2)}}{2}\right)\nabla_q^{-1} f(t/q). \tag{105}$$

Example 4.9 Applying $f(t) = t$, $t = 6$, $q = 7$, & $m = 2$ in the equation (105), we get

$$\boldsymbol{LHS} = \sum_{r=0}^{m}\left(\frac{(r+3)^{(3)}}{3^{(2)}}\right) f(tq^{m-r})$$
$$= \sum_{r=0}^{2}\left(\frac{(r+3)^{(3)}}{3^{(2)}}\right)(tq^{2-r})$$
$$= \left(\frac{3^{(3)}}{3^{(2)}}\right)(tq^2) + \left(\frac{4^{(3)}}{3^{(2)}}\right)(tq^1) + \left(\frac{5^{(3)}}{3^{(2)}}\right)(t)$$
$$= 522.$$

We know that,

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$\nabla_q^{-r} t = t\left(\frac{q}{q-1}\right)^r$, where $r = 1,2,3,4$.

$$\boldsymbol{RHS} = \nabla_q^{-4} f(tq^m) - \nabla_q^{-4} f(t/q) - (m+1)\nabla_q^{-3} f(t/q) - \left(\frac{(m+2)^{(2)}}{2}\right)\nabla_q^{-2} f(t/q) - \left(\frac{(m+3)^{(2)}}{2}\right)\nabla_q^{-1} f(t/q) =$$

$$(tq^m)\left(\frac{q}{q-1}\right)^4 - (t/q)\left(\frac{q}{q-1}\right)^4 - (3)(t/q)t\left(\frac{q}{q-1}\right)^3 - \left(\frac{(4)(3)}{2}\right)(t/q)\left(\frac{q}{q-1}\right)^2 - \left(\frac{(5)(4)}{2}\right)(t/q)\left(\frac{q}{q-1}\right)$$

$$= (6)(7^2)\left(\frac{7}{6}\right)^4 - (6/7)\left(\frac{7}{6}\right)^4 - (3)(6/7)\left(\frac{7}{6}\right)^3 - \left(\frac{(4)(3)}{2}\right)(6/7)\left(\frac{7}{6}\right)^2 - \left(\frac{(5)(4)}{2}\right)(6/7)\left(\frac{7}{6}\right)$$

$$= 522.$$

$$\therefore LHS = RHS$$

Theorem 4.10 If f be a real valued function and ∇_q^{-r} where $r = 1,2,\cdots,n$exists, then for every positive integer m,

$$\sum_{r=0}^{m}\left(\frac{(r+(n-1))^{(n-1)}}{(n-1)!}\right) f(tq^{m-r}) = \left\{\nabla_q^{-n} f(tq^m) - \nabla_q^{-n} f(t/q) - \left(\frac{(m+1)^{(1)}}{1!}\right)\nabla_q^{-(n-1)} f(t/q) - \left(\frac{(m+2)^{(2)}}{2!}\right)\nabla_q^{-(n-2)} f(t/q) - \cdots - \left(\frac{(m+(n-1))^{(n-1)}}{(n-1)!}\right)\nabla_q^{-1} f(t/q)\right\}. \quad (106)$$

Proof: For $n = 1$, in the equation (106), which gives

$$f(tq^m) + f(tq^{m-1}) + f(tq^{m-2}) + \cdots + f(t) = \nabla_q^{-1} f(tq^{m+1}) - \nabla_q^{-1} f(t)$$

For $n = 2$, in the equation (106), which gives

$f(tq^m) + 2f(tq^{m-1}) + \cdots + (m+1)f(t)$
$= \left\{\nabla_q^{-2} f(tq^m) - \nabla_q^{-2} f(t/q) - (m+1)\nabla_q^{-1} f(t/q)\right\}$ For $n = 3$, in the equation (106), which gives

$$\sum_{r=0}^{m}\frac{(r+2)^{(2)}}{2}f(tq^{m-r})=\nabla_q^{-3}f(tq^m)-\nabla_q^{-3}f(t/q)-(m+1)\nabla_q^{-2}f(t/q)-\frac{(m+2)^{(2)}}{2}\nabla_q^{-1}f(t/q)$$

For $n=4$, in the equation (106), which gives

$$\sum_{r=0}^{m}\frac{(r+3)^{(3)}}{3(2)}f(tq^{m-r})=\nabla_q^{-4}f(tq^m)-\nabla_q^{-4}f(t/q)-(m+1)\nabla_q^{-3}f(t/q)-\left(\frac{(m+2)^{(2)}}{2}\right)\nabla_q^{-2}f(t/q)$$

Proceeding this upto n times, we get

$$\therefore \sum_{r=0}^{m}\left(\frac{(r+(n-1))^{(n-1)}}{(n-1)!}\right)f(tq^{m-r})$$
$$=\left\{\nabla_q^{-n}f(tq^m)-\nabla_q^{-n}f(t/q)-\left(\frac{(m+1)^{(1)}}{1!}\right)\nabla_q^{-(n-1)}f(t/q)-\left(\frac{(m+2)^{(2)}}{2!}\right)\right.$$
$$\left.\nabla_q^{-(n-2)}f(t/q)-\cdots-\left(\frac{(m+(n-1))^{(n-1)}}{(n-1)!}\right)\nabla_q^{-1}f(t/q)\right\}.$$

Thus, we have obtained the inverse of q-Nabla Operator and some basic results using these operator and these results are validated with suitable examples.

5 $\alpha(q)$-Nabla operator

In this section, we explore the extend Nabla Operator with shift value $\alpha(q)$ which gives the summation formulae related to these operator and also we explore the inverse of these operator.

Definition 5.1 Let $f(t)$ be a real valued function and t, q, $\alpha \neq 0$ be a fixed real numbers. Then, the $\alpha(q)$-Nabla operator is defined as

$$\nabla_{\alpha(q)}f(t)=f(tq)-\alpha f(t/q) \tag{107}$$

and the inverse of the $\alpha(q)$-Nabla operator $\nabla_{\alpha(q)}^{-1}$ is defined as

$$f(t)=\nabla_{\alpha(q)}g(t) \implies \nabla_{\alpha(q)}^{-1}f(t)=g(t)+c$$

where c is an arbitrary constant.

Theorem 5.2 If f be a real valued function, $\alpha \neq 0$, $\nabla_{\alpha(q)}^{-1}f(t)$ exists, then for every positive integer m,
$f(tq^m)+\alpha f(tq^{m-1})+\alpha^2 f(tq^{m-2})+\cdots+\alpha^m f(t)=\nabla_{\alpha(q)}^{-1}f(tq^m)-\alpha^{m+1}\nabla_{\alpha(q)}^{-1}f(t)$.

Proof: From the equation (107),we have $\nabla_{\alpha(q)}f(t) = f(t) - \alpha f(t/q)$

Let $\nabla_{\alpha(q)^{-1}}f(t) = g(t)$

$$\implies f(t) = g(t) - \alpha g(t/q)$$

$$f(t) + \alpha g(t/q) = g(t) \tag{108}$$

Replace t by tq in the above equation,

$$f(tq) + \alpha g(t) = g(tq)$$

Substitute (108) in above equation,

$$f(tq) + \alpha f(t) + \alpha^2 g(t/q) = g(tq)$$

Again replace t by tq in the above equation,

$$f(tq^2) + \alpha f(tq) + \alpha^2 g(t) = g(tq^2)$$

Substitute (108) in above equation,

$$f(tq^2) + \alpha f(tq) + \alpha^2 f(t) + \alpha^3 g(t/q) = g(tq^2)$$

In general, $f(tq^m) + \alpha f(tq^{m-1}) + \alpha^2 f(tq^{m-2}) + \cdots + \alpha^m f(t) + \alpha^{m+1} g(t/q) = g(tq^m)$

$$\therefore f(tq^m) + \alpha f(tq^{m-1}) + \cdots + \alpha^m f(t) = \nabla^{-1}_{\alpha(q)} f(tq^m) - \alpha^{m+1} \nabla^{-1}_{\alpha(q)} f(t/q). \tag{109}$$

Example 5.3 Applying $f(t) = t$, $t = 3$, $\alpha = 2$, $q = 4$, $m = 3$ in equation (109), we get

$\boldsymbol{LHS} = f(tq^m) + \alpha f(tq^{m-1}) + \cdots + \alpha^m f(t)$

$= (tq^m) + \alpha(tq^{m-1}) + \alpha^2(tq^{m-2}) + \alpha^3(tq^{m-3})$

$= (3)(4^3) + (2)(3)(4^2) + (2^2)(3)(4) + (2^3)(3) = 360.$

We know that, $\nabla_{\alpha(q)}t = t - \alpha\,(t/q) = t\left(\dfrac{q-\alpha}{q}\right)$, $\nabla^{-1}_{\alpha(q)}t = t\left(\dfrac{q}{q-\alpha}\right)$

$\boldsymbol{RHS} = \nabla^{-1}_{\alpha(q)} f(tq^m) - \alpha^{m+1} \nabla^{-1}_{\alpha(q)} f(t/q)$

$= (tq^m)\left(\dfrac{q}{q-\alpha}\right) - \alpha^{m+1}\left(\dfrac{t}{q}\right)\left(\dfrac{q}{q-\alpha}\right)$

$= (3)(4^3)\left(\dfrac{4}{2}\right) - (2^4)\left(\dfrac{3}{4}\right)\left(\dfrac{4}{2}\right)$

$= 360.$

Theorem 5.4 If f be a real valued function and $\alpha \neq 0$, $\nabla^{-r}_{\alpha(q)} f(t)$ where $r = 1,2$

exists, then for every positive integer m,

$$f(tq^m)+2\alpha f(tq^{m-1})+3\alpha^2 f(tq^{m-2})+\cdots+(m+1)\alpha^m f(t) \\ =\nabla^{-2}_{\alpha(q)}f(tq^m)-\alpha^{m+1}\nabla^{-2}_{\alpha(q)}f(t/q)-(m+1)\alpha^{m+1}\nabla^{-1}_{\alpha(q)}f(t/q).$$

Proof: Consider,

$$f(tq^m)+\alpha f(tq^{m-1})+\cdots+\alpha^m f(t)=\nabla^{-1}_{\alpha(q)}f(tq^m)-\alpha^{m+1}\nabla^{-1}_{\alpha(q)}f(t/q) \tag{110}$$

Suppose $m=4,3,2,1$ and 0 in the equation (110), we get

$$f(tq^4)+\alpha f(tq^3)+\alpha^2 f(tq^2)+\alpha^3 f(tq)+\alpha^4 f(t)=\nabla^{-1}_{\alpha(q)}f(tq^4)-\alpha^5\nabla^{-1}_{\alpha(q)}f(t/q) \tag{111}$$

$$f(tq^3)+\alpha f(tq^2)+\alpha^2 f(tq)+\alpha^3 f(t)=\nabla^{-1}_{\alpha(q)}f(tq^3)-\alpha^4\nabla^{-1}_{\alpha(q)}f(t/q) \tag{112}$$

$$f(tq^2)+\alpha f(tq)+\alpha^2 f(t)=\nabla^{-1}_{\alpha(q)}f(tq^2)-\alpha^3\nabla^{-1}_{\alpha(q)}f(t/q) \tag{113}$$

$$f(tq)+\alpha f(t)=\nabla^{-1}_{\alpha(q)}f(tq)-\alpha^2\nabla^{-1}_{\alpha(q)}f(t/q) \tag{114}$$

$$f(t)=\nabla^{-1}_{\alpha(q)}f(t)-\alpha\nabla^{-1}_{\alpha(q)}f(t/q) \tag{115}$$

Multiply by α, α^2, α^3, α^4 in the equation (112), (113), (114) & (115) respectively,

$$\alpha f(tq^3)+\alpha^2 f(tq^2)+\alpha^3 f(tq)+\alpha^4 f(t)=\alpha\nabla^{-1}_{\alpha(q)}f(tq^3)-\alpha^5\nabla^{-1}_{\alpha(q)}f(t/q) \tag{116}$$

$$\alpha^2 f(tq^2)+\alpha^3 f(tq)+\alpha^4 f(t)=\alpha^2\nabla^{-1}_{\alpha(q)}f(tq^2)-\alpha^5\nabla^{-1}_{\alpha(q)}f(t/q) \tag{117}$$

$$\alpha^3 f(tq)+\alpha^4 f(t)=\alpha^3\nabla^{-1}_{\alpha(q)}f(tq)-\alpha^5\nabla^{-1}_{\alpha(q)}f(t/q) \tag{118}$$

$$\alpha^4 f(t)=\alpha^4\nabla^{-1}_{\alpha(q)}f(t)-\alpha^5\nabla^{-1}_{\alpha(q)}f(t/q) \tag{119}$$

Adding the equations (111),(116) to (119), we get

$$f(tq^4)+2\alpha f(tq^3)+3\alpha^2 f(tq^2)+4\alpha^3 f(tq)+5\alpha^4 f(t) \\ =\nabla^{-2}_{\alpha(q)}f(tq^4)-\alpha^5\nabla^{-2}_{\alpha(q)}f(t/q)-5\alpha^5\nabla^{-1}_{\alpha(q)}f(t/q)$$

In general,

$$\therefore f(tq^m)+2\alpha f(tq^{m-1})+3\alpha^2 f(tq^{m-2})+\cdots+(m+1)\alpha^m f(t)$$

$$=\nabla^{-2}_{\alpha(q)}f(tq^m)-\alpha^{m+1}\nabla^{-2}_{\alpha(q)}f(t/q)-(m+1)\alpha^{m+1}\nabla^{-1}_{\alpha(q)}f(t/q). \tag{120}$$

Example 5.5 Applying $f(t)=t$, $t=2$, $q=3$, $m=2$, $\alpha=2$ in the equation (120), we get

$$\boldsymbol{LHS} = f(tq^m) + 2\alpha f(tq^{m-1}) + \cdots + (m+1)\alpha^m f(t)$$
$$= (tq^2) + 2\alpha(tq^1) + 3\alpha^2(tq^0)$$
$$\therefore f(tq^m) + 2\alpha f(tq^{m-1}) + \cdots + (m+1)\alpha^m f(t) = 66.$$

We know that, $\nabla^{-1}_{\alpha(q)} t = t\left(\frac{q}{q-\alpha}\right)$, $\nabla^{-2}_{\alpha(q)} t = t\left(\frac{q}{q-\alpha}\right)^2$

$$\boldsymbol{RHS} = \nabla^{-2}_{\alpha(q)} f(tq^m) - \alpha^{m+1}\nabla^{-2}_{\alpha(q)} f(t/q) - (m+1)\alpha^{m+1}\nabla^{-1}_{\alpha(q)} f(t/q)$$
$$= \left(\frac{q}{q-\alpha}\right)^2 (tq^m) - \alpha^3 \left(\frac{q}{q-\alpha}\right)^2 (t/q) - (3)(\alpha^3)\left(\frac{q}{q-\alpha}\right)(t/q)$$
$$= \left(\frac{3}{3-2}\right)^2 (2)(3^2) - (2)^3 \left(\frac{3}{3-2}\right)^2 (2/3) - (3)(2^3)\left(\frac{3}{3-2}\right)(2/3) = 66.$$

$$\therefore LHS = RHS$$

Theorem 5.6 If f be a real valued function and $\alpha \neq 0$, $\nabla^{-r}_{\alpha(q)} f(t)$ where $r = 1,2,3$ exists, then for every positive integer m,

$$\sum_{r=0}^{m} \alpha^r \left(\frac{(r+2)^{(2)}}{2}\right) f(tq^{m-r}) = \nabla^{-3}_{\alpha(q)} f(tq^m) - \alpha^{m+1}\nabla^{-3}_{\alpha(q)} f(t/q) - (m+1)\alpha^{m+1}\nabla^{-2}_{\alpha(q)} f(t/q) - \alpha^{m+1}\left(\frac{(m+2)^{(2)}}{2}\right)\nabla^{-1}_{\alpha(q)} f(t/q).$$

Proof: Consider,

$$f(tq^m) + 2\alpha f(tq^{m-1}) + \cdots + \alpha^m (m+1) f(t)$$
$$= \nabla^{-2}_{q} f(tq^m) - \alpha^{m+1}\nabla^{-2}_{q} f(t/q) - (m+1)\alpha^{m+1}\nabla^{-1}_{\alpha(q)} f(t/q) \quad (121)$$

Applying $\nabla^{-1}_{\alpha(q)}$ in the equation (109),

$$\nabla^{-2}_{\alpha(q)} f(tq^m) + \nabla^{-2}_{\alpha(q)} f(tq^{m-1}) + \cdots + \nabla^{-2}_{\alpha(q)} f(t) = \nabla^{-3}_{\alpha(q)} f(tq^m) - \alpha^{m+1}\nabla^{-3}_{\alpha(q)} f(t/q)$$

Suppose $m = 4,3,2,1$ and 0 in the equation (121), we get

$$f(tq^4) + 2\alpha f(tq^3) + 3\alpha^2 f(tq^2) + 4\alpha^3 f(tq) + 5\alpha^4 f(t)$$
$$= \nabla^{-2}_{\alpha(q)} f(tq^4) - \alpha^5\nabla^{-2}_{\alpha(q)} f(t/q) - 5\alpha^5\nabla^{-1}_{\alpha(q)} f(t/q) \quad (122)$$

$$f(tq^3)+2\alpha f(tq^2)+3\alpha^2 f(tq)+4\alpha^3 f(t)=\nabla^{-2}_{\alpha(q)}f(tq^3)-\alpha^4\nabla^{-2}_{\alpha(q)}f(t/q)-4\alpha^4\nabla^{-1}_{\alpha(q)}f(t/q) \tag{123}$$

$$f(tq^2)+2\alpha f(tq)+3\alpha^2 f(t)=\nabla^{-2}_{\alpha(q)}f(tq^2)-\alpha^3\nabla^{-2}_{\alpha(q)}f(t/q)-3\alpha^3\nabla^{-1}_{\alpha(q)}f(t/q) \tag{124}$$

$$f(tq)+2\alpha f(t)=\nabla^{-2}_{\alpha(q)}f(tq)-\alpha^2\nabla^{-2}_{\alpha(q)}f(t/q)-2\alpha^2\nabla^{-1}_{\alpha(q)}f(t/q) \tag{125}$$

$$f(t)=\nabla^{-2}_{\alpha(q)}f(t)-\alpha\nabla^{-2}_{\alpha(q)}f(t/q)-\alpha\nabla^{-1}_{\alpha(q)}f(t/q) \tag{126}$$

Multiply by α, α^2, α^3 & α^4 in the equation (123), (124), (125), (126) respectively,

$$\alpha f(tq^3)+2\alpha^2 f(tq^2)+3\alpha^3 f(tq)+4\alpha^4 f(t)=\alpha\nabla^{-2}_{\alpha(q)}f(tq^3)-\alpha^5\nabla^{-2}_{\alpha(q)}f(t/q)-4\alpha^5\nabla^{-1}_{\alpha(q)}f(t/q) \tag{127}$$

$$\alpha^2 f(tq^2)+2\alpha^3 f(tq)+3\alpha^4 f(t)=\alpha^2\nabla^{-2}_{\alpha(q)}f(tq^2)-\alpha^5\nabla^{-2}_{\alpha(q)}f(t/q)-3\alpha^5\nabla^{-1}_{\alpha(q)}f(t/q) \tag{128}$$

$$\alpha^3 f(tq)+2\alpha^4 f(t)=\alpha^3\nabla^{-2}_{\alpha(q)}f(tq)-\alpha^5\nabla^{-2}_{\alpha(q)}f(t/q)-2\alpha^5\nabla^{-1}_{\alpha(q)}f(t/q) \tag{129}$$

$$\alpha^4 f(t)=\alpha^4\nabla^{-2}_{\alpha(q)}f(t)-\alpha^5\nabla^{-2}_{\alpha(q)}f(t/q)-\alpha^5\nabla^{-1}_{\alpha(q)}f(t/q) \tag{130}$$

Adding the equations (122), (127) to (130), we get

$$\textbf{LHS}=\left(\frac{2^{(2)}}{2}\right)f(tq^4)+\alpha\left(\frac{3^{(2)}}{2}\right)f(tq^3)+\alpha^2\left(\frac{4^{(2)}}{2}\right)f(tq^2)+\alpha^3\left(\frac{5^{(2)}}{2}\right)f(tq^1)+\alpha^4\left(\frac{6^{(2)}}{2}\right)f(t)$$

$$\therefore\sum_{r=0}^{4}\alpha^r\left(\frac{(r+2)^{(2)}}{2}\right)f(tq^{4-r})=\Big\{\nabla^{-3}_{\alpha(q)}f(tq^4)-\alpha^5\nabla^{-3}_{\alpha(q)}f(t/q)-5\alpha^5\nabla^{-2}_{\alpha(q)}f(t/q)-\left(\frac{6^{(2)}}{2}\right)\alpha^5\nabla^{-1}_{\alpha(q)}f(t/q)\Big\}$$

In general,

$$\therefore\sum_{r=0}^{m}\alpha^r\left(\frac{(r+2)^{(2)}}{2}\right)f(tq^{m-r})=\nabla^{-3}_{\alpha(q)}f(tq^m)-\alpha^{m+1}\nabla^{-3}_{\alpha(q)}f(t/q)-(m+1)\alpha^{m+1}\nabla^{-2}_{\alpha(q)}f(t)-\left(\frac{(m+2)^{(2)}}{2}\right)\alpha^{m+1}\nabla^{-1}_{\alpha(q)}f(t) \tag{131}$$

Example 5.7 Applying $f(t)=t,\ t=3,\ q=6,\ m=2,\ \alpha=2$ in the equation (131), we get

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$$\boldsymbol{LHS} = \sum_{r=0}^{m} \alpha^r \left(\frac{(r+2)^{(2)}}{2} \right) f(tq^{m-r})$$
$$= \sum_{r=0}^{2} \alpha^r \left(\frac{(r+2)^{(2)}}{2} \right) (tq^{2-r})$$
$$= \alpha^0 \left(\frac{(2)^{(2)}}{2} \right) (3)(6^2) + \alpha^1 \left(\frac{(3)^{(2)}}{2} \right) (3)(6) + (2^2) \left(\frac{(4)^{(2)}}{2} \right) (3)$$
$$= 288.$$

We know that, $\nabla_{\alpha(q)}^{-r} t = t \left(\frac{q}{q-\alpha} \right)^r$ where $r = 1,2,3$

$$= \nabla_{\alpha(q)}^{-3} f(tq^m) - \alpha^{m+1} \nabla_{\alpha(q)}^{-3} f(t/q) - (m+1)\alpha^{m+1} \nabla_{\alpha(q)}^{-2} f(t) - \left(\frac{(m+3)^{(2)}}{2} \right) \alpha^{m+1} \nabla_{\alpha(q)}^{-1} f(t)$$
$$= \left\{ \left(\frac{q}{q-\alpha} \right)^3 (tq^m) - \alpha^3 \left(\frac{q}{q-\alpha} \right)^3 (t/q) - (3)\alpha^3 \left(\frac{q}{q-\alpha} \right)^2 (t/q) \right.$$
$$\left. - \alpha^{m+1} \left(\frac{(m+3)^{(2)}}{2} \right) \left(\frac{q}{q-\alpha} \right) (t/q) \right\}$$
$$= \left\{ \left(\frac{6}{6-2} \right)^3 (3)(6^2) - 2^3 \left(\frac{6}{6-2} \right)^3 (3/6) - (3)(2)^3 \left(\frac{6}{6-2} \right)^2 (3/6) \right.$$
$$\left. - 2^3 \left(\frac{(4)^{(2)}}{2} \right) \left(\frac{6}{6-2} \right) (3/6) \right\}$$
$$= 288.$$
$$\therefore LHS = RHS$$

Theorem 5.8 If f be a real valued function, $\alpha \neq 0$ and $\nabla_{\alpha(q)}^{-r} f(t)$ where $r = 1,2,3,4$ exists, then for every positive integer m,

$$\sum_{r=0}^{m} \alpha^r \left(\frac{(r+3)^{(3)}}{3^{(2)}} \right) f(tq^{m-r}) = \nabla_{\alpha(q)}^{-4} f(tq^m) - \alpha^{m+1} \nabla_{\alpha(q)}^{-4} f(t/q) - (m+1)\alpha^{m+1} \nabla_{\alpha(q)}^{-3} f(t/q)$$
$$- \left(\frac{(m+2)^{(2)}}{2} \right) \alpha^{m+1} \nabla_{\alpha(q)}^{-2} f(t/q) - \left(\frac{(m+3)^{(3)}}{3^{(2)}} \right) \alpha^{m+1} \nabla_{\alpha(q)}^{-1} f(t/q) \qquad (132)$$

Example 5.9 Applying $f(t) = t$, $t = 8$, $\alpha = 4$, $q = 6$ & $m = 3$ in the equation (132),

$$\boldsymbol{LHS} = \sum_{r=0}^{m} \alpha^r \left[\frac{(r+3)^{(3)}}{3^{(2)}} \right] f(tq^{m-r})$$

$$= \sum_{r=0}^{3} (4)^r \left(\frac{(r+3)^{(3)}}{3^{(2)}} \right) (tq^{3-r})$$

$$= \left(4^{(0)} \left(\frac{3^{(3)}}{3^{(2)}} \right) (8)(6^3) + 4^{(1)} \left(\frac{4^{(3)}}{3^{(2)}} \right) (8)(6^2) + 4^{(2)} \left(\frac{5^{(3)}}{3^{(2)}} \right) (8)(6) + 4^{(3)} \left(\frac{6^{(3)}}{3^{(2)}} \right) (8) \right)$$

$$= 24256.$$

We know that, $\nabla_{\alpha(q)}^{-r} t = t \left(\frac{q}{q-\alpha} \right)^r$, where $r = 1,2,3,4$.

$$\boldsymbol{RHS} = \Bigg\{ \left(\frac{q}{q-\alpha} \right)^4 (tq^m) - \alpha^4 \left(\frac{q}{q-\alpha} \right)^4 (t/q) - (4)\alpha^4 \left(\frac{q}{q-\alpha} \right)^3 (t/q)$$

$$-\alpha^{m+1} \left(\frac{(m+2)^{(2)}}{2} \right) \left(\frac{q}{q-\alpha} \right)^2 (t/q) - \left(\frac{(m+3)^{(3)}}{3^{(2)}} \right) \alpha^{m+1} \left(\frac{q}{q-\alpha} \right) (t/q) \Bigg\}$$

$$= \Bigg\{ \left(\frac{6}{2} \right)^4 (8)(6^3) - 4^4 \left(\frac{6}{2} \right)^4 (8/6) - (4)(4)^4 \left(\frac{6}{2} \right)^3 (8/6)$$

$$-4^4 \left(\frac{(5)^{(2)}}{2} \right) \left(\frac{6}{2} \right)^2 (8/6) - 4^4 \left(\frac{(6)^{(3)}}{3^{(2)}} \right) \left(\frac{6}{2} \right) (8/6) \Bigg\} = 24256.$$

$$\therefore LHS = RHS$$

Theorem 5.10 If f be a real valued function and $\alpha \neq 0$, $\Delta_{\alpha(q)}^{-r} f(t)$ where $r = 1,2,\cdots,n$ exists then for every positive integer m,

$$\sum_{r=0}^{m} \alpha^r \left(\frac{(r+(n-1))^{(n-1)}}{(n-1)!} \right) f(tq^{m-r})$$

$$= \Bigg\{ \nabla_{\alpha(q)}^{-n} f(tq^m) - \alpha^{m+1} \nabla_{\alpha(q)}^{-n} f(t/q) - \left(\frac{(m+1)^{(1)}}{1!} \right) \alpha^{m+1} \nabla_{\alpha(q)}^{-(n-1)} f(t/q)$$

$$- \left(\frac{(m+2)^{(2)}}{2!} \right) \alpha^{m+1} \nabla_{\alpha(q)}^{-(n-2)} f(t/q) - \cdots - \left(\frac{(m+n-1)^{(n-1)}}{(n-1)!} \right) \alpha^{m+1} \nabla_{\alpha(q)}^{-1} f(t/q) \Bigg\} \tag{133}$$

Proof: For $n = 1$ in the equation (133), which gives

$$f(tq^m) + \alpha f(tq^{m-1}) + \cdots + \alpha^m f(t) = \nabla_{\alpha(q)}^{-1} f(tq^m) - \alpha^{m+1} \nabla_{\alpha(q)}^{-1} f(t)$$

For $n = 2$ in the equation (133), which gives

$$f(tq^m)+2\alpha f(tq^{m-1})+3\alpha^2 f(tq^{m-2})+\cdots+(m+1)\alpha^m f(t)$$
$$=\nabla^{-2}_{\alpha(q)}f(tq^m)-\alpha^{m+1}\nabla^{-2}_{\alpha(q)}f(t/q)-(m+1)\alpha^{m+1}\nabla^{-1}_{\alpha(q)}f(t/q)$$

For $n=3$ in the equation (133), which gives

$$\sum_{r=0}^{m}\alpha^r\left(\frac{(r+2)^{(2)}}{2}\right)f(tq^{m-r})=\nabla^{-3}_{\alpha(q)}f(tq^m)-\alpha^{m+1}\nabla^{-3}_{\alpha(q)}f(t/q)-(m+1)\alpha^{m+1}\nabla^{-2}_{\alpha(q)}f(t/q)$$
$$-\alpha^{m+1}\left(\frac{(m+2)^{(2)}}{2}\right)\nabla^{-1}_{\alpha(q)}f(t/q)$$

For $n=4$ in the equation (133), which gives

$$\sum_{r=0}^{m}\alpha^r\left[\frac{(r+3)^{(3)}}{3^{(2)}}\right]f(tq^{m-r})=\nabla^{-4}_{\alpha(q)}f(tq^m)-\alpha^{m+1}\nabla^{-4}_{\alpha(q)}f(t/q)-(m+1)\alpha^{m+1}\nabla^{-3}_{\alpha(q)}f(t/q)$$
$$-\left(\frac{(m+2)^{(2)}}{2}\right)\alpha^{m+1}\nabla^{-2}_{\alpha(q)}f(t/q)-\left(\frac{(m+3)^{(3)}}{3^{(2)}}\right)\alpha^{m+1}\nabla^{-1}_{\alpha(q)}f(t/q)$$

Proceeding this upto n times, we get

$$\therefore\sum_{r=0}^{m}\alpha^r\left(\frac{(r+(n-1))^{(n-1)}}{(n-1)!}\right)f(tq^{m-r})$$
$$=\Bigg\{\nabla^{-n}_{\alpha(q)}f(tq^m)-\alpha^{m+1}\nabla^{-n}_{\alpha(q)}f(t/q)-\left(\frac{(m+1)^{(1)}}{1!}\right)\alpha^{m+1}\nabla^{-(n-1)}_{\alpha(q)}f(t/q)$$
$$-\left(\frac{(m+2)^{(2)}}{2!}\right)\alpha^{m+1}\nabla^{-(n-2)}_{\alpha(q)}f(t/q)-\cdots-\left(\frac{(m+n-1)^{(n-1)}}{(n-1)!}\right)\alpha^{m+1}\nabla^{-1}_{\alpha(q)}f(t/q)\Bigg\}.$$

Thus, we have obtained Nabla operator with shift value $\alpha(q)$ and its inverse related to these operator. Also, Our results are validated with suitable examples.

6 Conclusion

In this project, we developed fundamental theorems using q-Delta, $\alpha(q)$-Delta, q-Nabla and $\alpha(q)$-Nabla Operator. Also we arrived at several fundamental theorems on integer order Delta and Nabla integration. Also, we explored the First order Fibonacci sequence through the Nabla operator, with a focus on fundamental theorems. Nabla operator provides a powerful tool for analyzing the first order Fibonacci sequence. Furthermore, our investigation revealed that the Nabla operator can be used to establish connections between the Fibonacci sequence and the other areas of mathematics, such as difference equation and discrete mathematics. Our results may have implications for various fields including mathematics, computer science and data analysis. Our results are validated with suitable numerical examples.

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