2456-8686, xi(i),2025:167-176
https://doi.org/10.26524/cm209

ORIGINAL RESEARCH **Open Access**

The Non-Autonomous Saddle Node Bifurcation

Geethalakshmi S[1], Britto Antony Xavier [2] and Divya Bahrathi S[3]

Received: 10 May 2025/ Accepted: 26 May 2025/ Published online: 16 June 2025

Abstract

This work explores the phenomenon of saddle-node bifurcation in both autonomous and non-autonomous dynamical systems. The classical autonomous case is illustrated through a simple differential equation that demonstrates the creation and annihilation of fixed points as a system parameter varies. The focus then shifts to the more complex non-autonomous case, where the system's behavior depends explicitly on time. It is shown that, under certain conditions, a similar bifurcation occurs even without time-invariant dynamics.

Key Words: Bifurcation , Saddle-node Bifurcation, Non-autonomous , Trajectory.

AMS Classification: 34C23, 34C37, 34D05, 34A34.

1 Introduction

The canonical example of an autonomous equation in which a saddle-node bifurcation occurs is,

$$\dot{x} = \lambda - x^2 \tag{1.1}$$

For $\lambda < 0$ every trajectory tends to $-\infty$,

For $\lambda > 0$ there are two fixed points: a stable point at $x\sqrt{\lambda}$ and an unstable point at $x = -\sqrt{\lambda}$

[1] Department of Mathematics, Department of Mathematics, Sacred Heart College (Autonomous),Tirupattur, Tamil Nadu, South India. Email: geethathiru126@gmail.com
[2] Department of Mathematics, Department of Mathematics, Sacred Heart College (Autonomous),Tirupattur, Tamil Nadu, South India. Email: brittoshc@gmail.com
[3] Department of Mathematics, Department of Mathematics, Sacred Heart College (Autonomous),Tirupattur, Tamil Nadu, South India. Email: divyabharathitacw@gmail.com

In the non-autonomous case the equation,

$$\dot{x} = f(x, t, \lambda) \tag{1.2}$$

undergoes a local saddle node bifurcation at $x = 0$, $\lambda = 0$ provided that there exists a $\lambda_0 > 0$ and $c > 0$, a δ with $0 < \delta < c$ such that,

(i) For $-\lambda_0 < \lambda \leq 0$ there are no complete trajectories lying within $(-\epsilon, \epsilon)$

(ii) For $0 < \lambda < \lambda_0$ there exists a complete trajectory $x_\lambda{}^{+}(.)$ that it attract within $(-\delta, \epsilon)$ and another complete trajectory $x_\lambda{}^{-}(.)$ that lies within $(-\epsilon, \epsilon)$ and is asymptotically unstable.

$$\Rightarrow \lim_{\lambda\to 0} x_\lambda{}^{\pm}(t) \to 0 \tag{1.3}$$

Uniformly on compact subintervals of $\mathbb{R}$.

2 GENERAL SADDLE NODE

We now consider, $\dot{x} = G(t, x, \lambda)$

$G(t, x, \lambda)$ we use taylor expansion where G are evaluated at $(t, 0,0)$

$$G(t, x, \lambda) = G + G_x x + G_\lambda \lambda + {}^1/_2\, G_{xx} x^2 + G_{x\lambda} x\lambda + {}^1/_2\, G_{\lambda\lambda}\lambda^2 + {}^1/_6\, G_{xxx} x^3 + {}^1/_3\, G_{xx\lambda} x^2\lambda + {}^1/_3\, G_{x\lambda\lambda} x\lambda^2 + {}^1/_6\, G_{\lambda\lambda\lambda}\lambda^3 + \cdots$$

Assume that $G(t, 0,0) = G_x(t, 0,0) = 0$ and we get,

$$G(t, x, \lambda) = \lambda\left[G_\lambda + G_{x\lambda} x + {}^1/_2\, G_{\lambda\lambda}\lambda + {}^1/_3\, G_{xx\lambda} x^2\lambda + {}^1/_3\, G_{x\lambda\lambda} x\lambda^2 + {}^1/_6\, G_{\lambda\lambda\lambda}\lambda^3 + \cdots\right] + \left[{}^1/_2\, G_{xx} + {}^1/_6\, G_{xxx} x + \cdots\right]x^2$$

2.1 Theorem:

Consider, $\dot{x} = G(t, x, \lambda)$

and assume that $G(t, 0,0) = G_x(t, o, o) = 0$, Set

$$f(l) = G_\lambda(l, 0,0) \text{ and } g(l) = -\tfrac{1}{2} G_{xx}(l, 0,0)$$

and rewrite the equation as,

$$\dot{x} = \lambda[f(t) + \phi(t, x, \lambda)] - x^2[g(t) + \psi(t, x)] \quad , \quad \text{where } \psi(t, 0) = 0.$$

Assume that,

$$\lim_{t\to\pm\infty} \inf g(t) > 0$$

(2.2.1)

and $0 < m = \lim_{t\to\pm\infty} inf \frac{f(t)}{g(t)} \le \lim_{t\to\pm\infty} sup \frac{f(t)}{g(t)} = M < +\infty$ (2.2.2)

and $|\phi(t, x, \lambda)| \le h(t)[|x| + |\lambda|]$

with $|\phi_x(t, x, \lambda)| \le h(t)$ (2.2.3)

and $|\psi(t, x)| \le h(t)$ where $\lim_{t\to\pm\infty} sup \frac{h(t)}{g(t)} \le k$

Then there is a local saddle node bifurcation as λ passes through zero. When $\lambda > 0$ the pullback attracting trajectory $x_\lambda(.)$ is forwards attracting in $(0, \epsilon)$, and the unstable trajectory is pullback repelling within $(-\epsilon, \delta)$.

Proof

A saddle-node bifurcation in the autonomous equation, $\dot{x} = f(x, \lambda)$ are,

$f(0,0) = 0, f_x(0,0) = 0, f_{x\lambda}(0,0) > 0, f_{xx}(0,0) < 0$

Using this conditions in,

$$G(t, x, \lambda) = f(x, \lambda)$$

The two assumptions on the x derivatives of ϕ and ψ imply Lipschitz bounds as,

$|\phi(t, x_1, \lambda) - \phi(t, x_2, \lambda)| \le h(t)|x_1 - x_2|$ and

$|\Psi(t, x_1) - \Psi(t, x_2)| \le h(t)|x_1 - x_2|$, for $\lambda < 0$ and C sufficiently small we have,
$\dot{x} \le \lambda[f(t) - \epsilon h(t)] \le \lambda f(t)[1 + (k\epsilon/m)]$, for $t \le -T$ (or) $t \ge T$,

There are no complete non zero trajectories that lie entirely within $(-\epsilon, \epsilon)$

When $\lambda = \delta^2$ we have, for all $|x| \le \epsilon$,

$\dot{x} \le g(t)[\delta^2(M + \epsilon k) - x^2(1 - \epsilon k)]$

and $\dot{x} \ge g(t)[\delta^2(m - \epsilon k) - x^2(1 + \epsilon k)]$

Then $\delta = \epsilon\sqrt{\frac{1-\epsilon k}{M+\epsilon k}}$ and it follow that any trajectory with,

$$x\_ = -\delta\sqrt{\frac{m-\epsilon k}{1+\epsilon k}} < x_\delta \le \epsilon$$

has $|x(t, s, x_{s)}| \le \epsilon$ for all $t \ge S$ and hence that

$$\delta\sqrt{\frac{m - \epsilon k}{1 + \epsilon k}} \le \lim_{\delta\to-\infty} x(t, s, x_s) \le \epsilon$$

Thus pullback attractor in $(x_-, \epsilon]$ consists of the interval $[x_1(t), x_2(t)]$.

Consider the difference $z = x_1 - x_2$ satisfies,

$$\frac{dz}{dt} = \lambda[\phi(t, x_1, \lambda) - \phi(t, x_2, \lambda)] - (x_1 + x_2)g(l)z - {x_1}^2\psi(t, x_1) + {x_2}^2\psi(t, x_{2)}$$

$$\frac{dz}{dt} \le \delta^2 h(t)z - (x_1 + x_2)g(t)z - ({x_2}^2 - {x_1}^2)\psi(t, x_1) + [\psi(t, x_{2)} - \psi(t, x_1)]{x_1}^2$$

$$\le \delta^2 h(t)z - (x_1 + x_2)\, g(t)z + [\epsilon + \delta^2]\, h(t)(x_1 + x_2)z + h(t)z\epsilon^2$$

$$\le C[\epsilon^2 h(t) - \epsilon g(t)]z$$

$$\le -C[1 - k\epsilon]\epsilon\, g(t)z$$

as $\epsilon \to 0$, for choosen ϵ, $z(t) = 0$.

This gives a saddle-node bifurcation.

3 The Special Case of Saddle-Node Bifurcation

3.1 Theorem

Consider the equation, $\dot{x} = \lambda f(t) - g(t)x^2$ (3.1.1)

Where f is essentially positive.

$$\int_{-\infty}^{t} f(s)ds = \int_{t}^{\infty} f(s)ds = +\infty \qquad (3.1.2)$$

and balance conditions,

$$\lim_{t\to\pm\infty} \inf g(t) > 0$$

and $0 < m \le \lim_{t\to\pm\infty} \frac{f(t)}{g(t)} \le m$ hold.

Then for $\lambda = 0$ there are no non-zero bounded complete trajectories.

When $\lambda < 0$ for any fixed x_s there is a σ such that,

for $S \le \sigma,\ x(t, s, x_s) \to -\infty as t \to t^*(s) < \infty$

Similarly for any fixed t we have,

$$x(t, s, x_s) \to -\infty\ as\ S \to S^*(t) > -\infty$$

For $\lambda = 0$, the zero solution is locally forwards and locally pullback attracting within $[0, \infty)$, while for negative initial conditions we have same behavior as for $\lambda < 0$.

For $\lambda > 0$ there are two trajectories $\pm x^*(t)$ such that $x^*(t)$ is both forwards and pullback attracting,

$$\lim_{S\to-\infty} S(t, s)x_0 = x^*(t), \text{for all } x_0 > -\sqrt{\lambda m} \text{ and}$$

$$\lim_{t\to+\infty} \text{dis t}(s(t, s)x_0, x^*(t)) = 0, \text{for all } x_0 > -\sqrt{\lambda m}$$

and $-x^*(t)$ is asymptotically unstable and pullback repelling,

$$\lim_{S\to+\infty} S(t,s)x_0 = -x^*(t), \text{for all } x_0 < \sqrt{\lambda m} \text{ and}$$

$$\lim_{t\to-\infty} \text{dis}\, t(s(t,s))x_0 - x^*(t)) = 0, \text{for all } x_0 < \sqrt{\lambda m}$$

Proof

We consider $\lambda < 0$ and assume initially that $x_s < 0$, Since there exists a T such that for $t \leq T$

The functions f and g are positive we have,

$$\dot{x} \leq -g(t)\, x^2 \quad \text{, for all t.}$$

$$x(t,s,x_s) \leq \frac{1}{x_s{}^{-1} + \int_s^t g(r)dr}, \forall\, S \leq t \leq T$$

Since g is essentially positive there exists an $s^*(t) > -\infty$ such that,

$$\lim_{s\to s^*(t)} x(t,s,x_s) = -\infty$$

Similarly if x_s is fixed there exists a $\sigma(t)$ such that if $S \leq \sigma(t)$ we have,

$$\lim_{t\to t^*(s)} x(t,s,x_s) = -\infty, \text{ for some } \ t^* < +\infty$$

If $x_s > 0$, the above results applied for $x_s = -1$, there exists aσ_1, such that if $S \leq \sigma_1$ then $x(t,s,-1) \to -\infty ast \to t^*(s) < \infty$

Since for $t \leq T$ we have $\dot{x} \leq \lambda f(t) < 0$ and

$$x(t,s,x_s) < x_s + \lambda \int_s^t f(r)dr \tag{3.1.3}$$

The essential positivity of f now implies that there exists a σ_2 such that if $S \leq \sigma_2$ then,

$$x(t,s,x_s) \leq -1 \text{ for some } t \leq \sigma_1$$

for some $t^*(s)$ we have $x(t,s,x_s) \to -\infty ast \to t^*(s)$

For each fixed t we have$x(t,s,-1) \to -\infty$ as $s \to s_1\,(t) > -\infty$

There exists an $s_2(t)$ such that$x(t,s,x_s) \to -1$ as $s \to s_2$.

It follows that $x(t,s,x_s) \to -\infty$ as $s \to s_1(s_2)$

When λ=0 the local attractivity of origin from explicit solution.

$$x(t,s,x_s) = \frac{1}{x_s{}^{-1} + \int_s^t g(r)dr}$$

For $x_s < 0$ is a consequence above for $\lambda < 0$.

when$\lambda > 0$ we have

$$\dot{x} = g(t)[M\lambda - x^2 \text{ and } \dot{x} \geq g(t)[m\lambda - x^2]$$

If $x_0 > -\sqrt{m\lambda}$ then,

$$\sqrt{m\lambda} \leq \lim_{s \to -\infty} x(t, s, x_0) \leq \sqrt{M\lambda}$$

Considering the difference of two solutions

$z = x_1 - x_2$we have,

$\frac{dz}{dt} = -g(t)[x_1 - x_2]z$

Since g is essentially positive and $x_1, x_2 \geq \sqrt{m\lambda}$ that $x_1(t) = x_2(t)$.

This gives a positive solution $x^*(t)$ that attracts (pullback and forwards) all trajectories with $x_0 > -\sqrt{\lambda m}$.

As $t \to +\infty$, for $x_0 < -\sqrt{\lambda m}$ the solution tends to $-\infty$. There is intermediate of conditions between $-\sqrt{\lambda M}$ and $-\sqrt{\lambda m}$.

Since f and g are symmetric in t.

There is negative solution $y^*(t)$ that attracts all trajectories with $x_0 < \sqrt{m\lambda}$ both backwards in time and is 'pullback repelling'

$$\lim_{t\to\infty} x(s, t, x_0) = y^*(s) \tag{3.1.4}$$

If $x_s > y^*(s)$ then,

$$\lim \quad _{t\to\infty}[x(s, t, x_s) - x^*(t)] = 0$$

Hence convergence holds.

If $x_s > -\sqrt{\lambda m}$

Now consider an initial condition ,$x_s > y^*(s)$

(3.1.4) holds for $x = 0$.

(i.e) $\lim_{t\to\infty}[x(t, s, 0) = y^*(s)]$,for t large enough we have, $x(s, t, 0) < x_s$

Since the equation is order-preserving that, $x(s, t, x_s) > 0$

From time t this solution is attracted to $x^*(t)$.

Reversing shows that $y^*(t)$ attracts any initial condition less than $x^*(t)$ as $t \to -\infty$.

3.2 Example For Saddle –Node Bifurcation

2456-8686, xi(i),2025:167-176
https://doi.org/10.26524/cm209

The simplest autonomous example which exhibits this bifurcation is,

$$\dot{x} = a - Bx^2$$

Clearly this equation has no fixed points for $a < 0$.

For $a > 0$ there are two fixed points $\pm\sqrt{a/B}$

If $x_0 < -\sqrt{a/B}$ then $x(t) \to -\infty$ in finite time,

If $x_0 > \sqrt{a/B}$ then $x(t, x_0) \to \sqrt{a/B}$ $as\ t \to +\infty$

We consider the non-autonomous equation,

$$\dot{x} = a - b(t)x^2 \tag{3.2.1}$$

Where $0 < b(t) \leq B, b(t) \to 0\ as\ |t| \to \infty$ and

$$\int_{-\infty}^{0} b(s)ds = \int_{0}^{\infty} b(s)S = \infty \tag{3.2.2}$$

The invariance of these conditions when $t \to -t$ greatly simplifies the analysis.

3.3 Saddle- Node Type Behaviour For $a \leq 0$

We consider $a \leq 0$ where the behavior is essentially the same as in the autonomous case.

3.4 Lemma

If $a < 0$ then every solution $S(t, s)x_0 \to -\infty$ in a finite time, both forwards (fix s and x_0 and let $t \to t^* < +\infty$) and pullback (fix t and x_0 and let $S \to S^* > -\infty$).

If $a = 0$ then solutions with $x_0 > 0$ tend to zero.

While if $x_0 < 0$ then $S(t, s)\ x_0$ tends to $-\infty$ in finite time (as in the case $a < 0$).

Proof

When $a = 0$, the equation can be solved explicitly to yield .

$$S(t, s)x_s = \frac{x_s}{1 + x_s \int_s^1 b(r)dr}$$

Hence Lemma is proved.

3.5 Saddle-Node Type Behaviour For $a > 0$

When $a > 0$ there are two complete trajectories which can identify as being stable and unstable.

3.6 Theorem

For $a > 0$ there exist two complex trajectories 3.2.1, $\alpha(t)$ and $\beta(t)$ which are bounded below by $\sqrt{a/B}$ and (respectively $-\sqrt{a/B}$) and diverge to $+\infty$(respectively$-\infty$) as $|t| \to \infty$. The solution $\alpha(t)$ is globally pullback asymptotically stable and the solution $\beta(t)$ is asymptotically unstable.

$$\lim_{s\to-\infty} x(t,s,x_0) = \alpha(t), \quad \forall x_0$$

$$\lim_{t\to-\infty}[x(t,s,x_0) - \beta(t)] = 0, \quad \forall x_0 < \alpha(S) \tag{3.6.1}$$

We have,

$$\lim_{t\to+\infty}[x(t,s,x_0) - \alpha(t)] = 0, \quad \forall x_0 > \beta(S)$$

$$\lim_{s\to+\infty} x(t,s,x_0) = \beta(t), \quad \forall x_0 \tag{3.6.2}$$

Proof

We consider the solutions $S(t,s)x_0$when $S \to -\infty$. If t and x_0 are fixed then there exists an $S_0(t,x_0)$ such that the solution is bounded below,

$$S(t,s)\, x_0\sqrt{a\,/\,B}\,, for\ all\ S \leq S_0 \tag{3.6.3}$$

Choose $\epsilon(x_0) < B$ small enough that $x_0 > -\sqrt{a/\epsilon}$.

Then there exists a time t(Є) such that $|b(t)| \leq \epsilon$ for all $t \leq t(\epsilon)$ and so,

$$\dot{x} \geq a - \epsilon x^2\ for\ all\ t \leq \tau(\epsilon)$$

There exists a time T such that the solution of,

$$\dot{y} = a - \epsilon y^2\,, \qquad t \geq T$$

If we take $S \leq S_0 \equiv \tau(\epsilon) - T$ we obtain (2.4.5)

Now, $\beta = inf_{r\epsilon[t-1,t]}\, b(r)$

On the time interval [t-1,t] we have,

$\dot{x} \leq a - \beta x^2$, for all $S \leq S_0 - 1$ the solution S (t,s)x_0 is also bounded above independently of value x_0, S_0 we have,

$$\sqrt{a/B} \leq S(t,s)x_0 \leq \sqrt{a/\beta} + \beta^{-1}\,, \text{for all } S \leq S_0 - 1$$

Then there exists a pullback attractor A(t), Which for each $t\epsilon\mathbb{R}$ is connected set.

Since A(t) is a subset of $\mathbb{R}$,

$A(t) = [\alpha(t), \alpha^+(t)]$,for some $\alpha(t) and\ \alpha^+(t)$

2456-8686, xi(i),2025:167-176
https://doi.org/10.26524/cm209

Since phase space is one dimensional, the process preserving and so $\alpha(t)$ *and* $\alpha^+(t)$ are trajectories of (3.6.2).

From (3.6.3) that $\alpha(t) \geq \sqrt{a/B}$ for all t

$$\alpha(t) \to +\infty \; as \; t \to -\infty \tag{3.6.4}$$

Consider time reversed $\tau - t, \alpha' = d\alpha/d\tau$ we have,

$$\alpha' = -a + b(-\tau)\ \alpha^2$$

Suppose (3.6.4) does not hold.

These exists a K such that for each τ_0 there exists a $\tau \geq \tau_0$ with $\alpha(t) \leq k$

Choose Є small enough that $k < \sqrt{a/\epsilon}$, and then T such that for all $t \geq T$, $b(-t) \leq \epsilon$, for all t we have ,

$$\alpha' \leq -a + \epsilon\alpha^2$$

Since $\alpha(t) \leq K < \sqrt{a/\epsilon}$ for some $t \geq T$,

$$\lim_{t\to\infty} \alpha(t) \leq -\sqrt{a/\epsilon} < \sqrt{a/B}$$

We obtain 3.6.4 by contradiction.

For all x_0 we have $x_0 < \alpha(s)$ for s small enough.

Since $S(t,s)$ is order preserving and $\alpha(t)$ is a complete trajectory. The pullback attraction property of A(t),

$$\lim_{S\to-\infty} dis\,[S(t,s)x_0, A(t)] = 0 \text{ we have,}$$

$$\lim \quad {}_{S\to-\infty} S(t,s)x_0 = \alpha(s)$$

$$\Rightarrow A^+(t) = \{\alpha(t)\}$$

Now consider the system with both time and space reversed $(y = -x \; and \; \tau = -t)$ the symmetry of conditions of bit to use exactly same arguments to prove the existence of complete trajectory $\beta(t)$ such that

$\beta(t) \leq -\sqrt{a/B}$, $\beta(t) \to -\infty \; as \; |t| \to \infty$and

$$\lim\ {}_{S\to\infty} S(t,s)x_0 = \beta(t), \;\; \forall\, x_0 \tag{3.6.5}$$

so that, $$Z(t) = |x_0 - y_0| \exp\left(-\int_s^t [x(r) + Y(r)\right] b(r)dr \tag{3.6.6}$$

From (3.2.2) that integral $\int_s^t b(r)dr$ diverges as $t \to +\infty$, any two trajectories are bounded by a positive quantity will converge as $t \to +\infty$

Taking $x_0 = \alpha(s)$ we know that $\alpha(t) \geq \sqrt{a/B}$, show that eventually $S(t,s)x_0 > 0$ for our choice of $x_0 > \beta(s)$.

(3.6.5) for t sufficiently large we have, $S(S,t)0 < x_0$ and the order- preserving property of $S(t,s) = [S(s,t)]^{-1}$ gives $0 < S(t,s)x_0$.

4 Conclusion

The study of the non-autonomous saddle-node bifurcation extends the classical understanding of bifurcation phenomena to systems with explicit time dependence. While the autonomous case presents fixed points whose stability changes with a parameter, the non-autonomous setting reveals more subtle dynamics involving entire trajectories rather than isolated equilibria.

References

[1] Arnold L, Random Dynamical systems, springer, Berlin, 1998.
[2] Buse O, Kuznetsov A, Perez R, Existence of limit cycles in the repressilator equation. Int.J.Bifurcation chaos, 19, 4097-4106(2009).
[3] Carr J, Applications of centre manifold theory, Appl. math.sci., vol.35, springer, berlin, 1981.
[4] Cheban DN, Kloeden PE, Schmalfub B, Pullback attractors in dissipative differential equations under discretization, J.Dynam. differential equations 13(1)185-213(2001).
[5] Edwards R, Glass L, Dynamics in genetic networks. The American mathematical monthly.121,793-809(2014).
[6] Fabbri R, Johnson RA , Mantellini F, A nonautonomous saddle-node Bifurcation pattern, Stoch. Dyn. 4(3), 335-350 (2004).
[7] Glendinning P , Non-smooth pitchfork bifurcations , Discrete contin. Dyn. Syst. B 4 (2),457-464(2004).
[8] Guillemil V, Pollack A, Differential topology.Prentice-Hall, 1974.
[9] Hairer E, Norsett SP, Wanner G, Solving ordinary differential equations I :Problems (second edition) Newyork : Springer-verlag, 1993.
[10] Hale JK, Ordinary differential equations , pure Appl. Math., vol. 21,krieger, Huntington,1980.
[11] Hassard BD, Kazarinoff ND, Wan Y-H, Theory and applications of Hopf bifurcation, Cambridge university press, Cambridge, 1981.
[12] Hirsch MV, Smale S, Differential equations, dynamical systems and linear algebra, Academic press,1974.
[13] Iooss G, Bifurcations of maps and applications, North-Holland, 1979.
[14] Johnson RA, Kloeden.P.E, Pavani.R, Two-step transition in nonautonomous bifurcations: An explanation, stoch.Dyn.2 (1)67-92(2002).
[15] Johnson RA, Yi Y, Hopf bifurcation from non-periodic solutions of differential equations , II, J. Differential Equations 107 (2) 310-340(1994).