

A Reliability Estimations of The Control System Using Cut And Tie Set Methods

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Abstract

The paper focuses on the challenge of assessing the reliability of control systems. Reliability evaluation at the system level is crucial for understanding overall system performance and guiding system adjustments in response to component failures. However, the dynamic and interconnected nature of control systems makes traditional reliability engineering methods inadequate. This paper proposes a novel approach that involves identifying equivalent tie sets or cut sets to evaluate reliability based on control system performance. This method offers the advantage of easily re-evaluating reliability when faults are detected or control objectives change by updating the cut sets or tie sets.

Key words: Network reliability, Reliable path, minimal paths (tie-sets).

AMS classification: 62N05,90B25.

1 Introduction

The paper addresses the challenge of quantifying reliability in fault-tolerant control systems. Unlike traditional reliability analysis, which focuses on series-parallel structures, control systems involve complex functional and dynamic relationships. Existing methods often neglect control objectives or dynamics, limiting their effectiveness in assessing system-level reliability. This paper proposes a novel approach that extends tie/cut-set methods to evaluate the reliability of control systems based on their performance requirements [9]. By relating system functions to control objectives, the proposed method can adapt to changes in operating conditions and performance requirements. The paper demonstrates the effectiveness of the approach through an example, highlighting the impact of loop gain changes on system

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reliability [2].

The paper introduces a novel method to assess the reliability of control systems. Unlike previous approaches that focus on individual components or neglect system dynamics, this method evaluates the overall system reliability [5]. The paper acknowledges that increasing component reliability doesn't always guarantee higher system reliability and that system reliability can be maintained despite component failures [6]. This approach aims to address the open problem of quantifying system-level reliability in fault-tolerant control systems. While previous studies have explored different methods, this paper presents a new approach that considers system dynamics and control objectives [4].

In this paper, we expand the tie/cut-set methodologies used for reliability analysis of networked systems to create a procedure for assessing control system reliability. In the suggested method, the system's required functions are tied to control performance or objectives. Furthermore, the technique is easily adaptable to changes in system component operating conditions and performance requirements. When such a change occurs, we just need to update the cut/tie set model and recalculate the reliability.

The rest of the paper is organised as follows: Chapter 2 provides a brief discussion of the fundamental principles of probability and reliability evaluation for network topologies; followed by an example to demonstrate the major operations. The results in the example demonstrate that simply modifying loop gains in the control system can result in changes to system reliability. Chapter 3 draws the conclusions.

2 Existing Techniques for Reliability Evaluation of Network Structures

A widely accepted definition of system reliability is as follows [6].

Definition 2.1 *Reliability*

The reliability, $R(t)$, of an item (a component or a system), is defined as the likelihood that, when operating under specific environmental conditions, it will execute its intended function satisfactorily within the specified time span $[0, t]$.

A reliability block diagram is a graphical representation of the link between system functionality and component functionality. In practice, a system is commonly depicted as a reliability block diagram in network topology, with components connected in series, parallel, mesh, or a combination of these. The cut/tie procedures can be used to determine their reliability [3]. It is worth noting

that in general, feedback does not exist in networks.

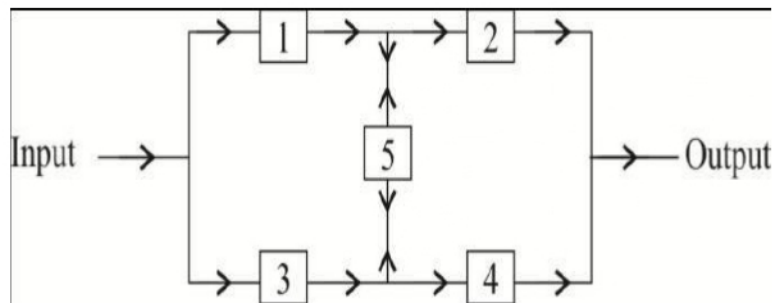


Figure 1: Bridge configuration.

2.1 Minimal Path Set

Let a system consist of n components, namely, $1, 2, 3, \dots, n$. Let C denote the set of all components of the system. Then $C = \{1, 2, 3, \dots, n\}$.

Definition 2.2 Path Set

A path set P is a subset of C such that if all the components of P are successful, then there exists a path between the input and output of the system.

Definition 2.3 Minimal Path Set

A path set P is said to be the minimal path set if there exists no proper subset of P as a path set in the system.

In other words, a path set P is said to be minimal path set if the failure of even a single component of P results in the failure of the path between the input and output.

For example, the path sets for the system having the reliability block diagram shown in Fig. 2 are as follows:

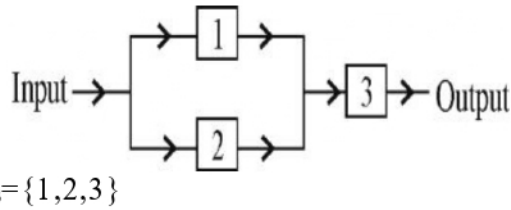


Figure 2: Reliability block diagram of a system having components 1 and 2 in parallel. The parallel configuration of 1 and 2 is in series with component 3.

Of these paths, P_1 and P_2 are minimal paths because there is no proper subset of P_1 and P_2 , which is also a path. But the path P_3 is not a minimal path because there exist paths P_1 and P_2 , which are proper subsets of P_3 .

2.2 Cut Set Method

Definition 2.4 *Cut Set*

A cut set is a set of system components which, when fail, cause failure of the entire system.

Definition 2.5 *Minimal Cut Set*

A minimal cut set is a set of system components which, when fail, cause failure of the entire system; but when any one component in the set does not fail, the system will not fail as a whole.

According to the definition, the system cannot fail unless all components of each cut set fail. As a result, the components in one cut set are effectively connected in parallel. Furthermore, if there are many cut sets, the system fails if all of the components in any of them fail. Consequently, all the cut sets are effectively connected in series. As a result, given a system/network, a basic parallel-series model can be constructed based on the reliability analysis cut sets.

Let a system consist of n components, namely, $1, 2, 3, \dots, n$. Let C denote the set of all components of the system. Then $C = \{1, 2, \dots, n\}$.

A **cut set** C_1 is a subset of C such that a failure of all components of C_1 results in the failure of the system.

A cut set C_1 is said to be the **minimal cut set** if there exists no proper subset of C_1 as a cut set in the system.

For example, the cut sets for the system having the reliability block diagram shown in Fig 2 are: $C_1 = \{3\}, C_2 = \{1, 3\}, C_3 = \{2, 3\}, C_4 = \{1, 2\}, C_5 = \{1, 2, 3\}$ of these cut sets, only C_1 and C_4 are minimal cut sets, because there exist no proper subsets of C_1 and C_4 , which are cut sets. For all other cut sets mentioned above, there exists at least one proper subset, which is a cut set. So these subsets are not minimal cut sets.

Algorithm:

In this method, we first find minimal cut sets for the given system. Let $C_1, C_2, C_3, \dots, C_k$ be the minimal cut sets for the system consisting of n components, namely, $1, 2, 3, \dots, n$. Let m_i denote the number of components in the cut set $C_i, i = 1, 2, 3, \dots, k$. By definition, all components of a minimal cut set must fail for the system to fail. This implies that the components of a cut set are connected in parallel from a reliability point of view. Further, the occurrence of any one cut set among $C_1, C_2, C_3, \dots, C_k$ results in the failure of the system. This implies that $C_1, C_2, C_3, \dots, C_k$ are connected in series from a reliability point of view. Thus, in terms of $C_1, C_2, C_3, \dots, C_k$, The reliability block diagram of the system would be as shown in Fig.3,

where the m_i components of cut set C_i are denoted by $C_{i1}, C_{i2}, \dots, C_{im_i}, 1 \leq i \leq k$.

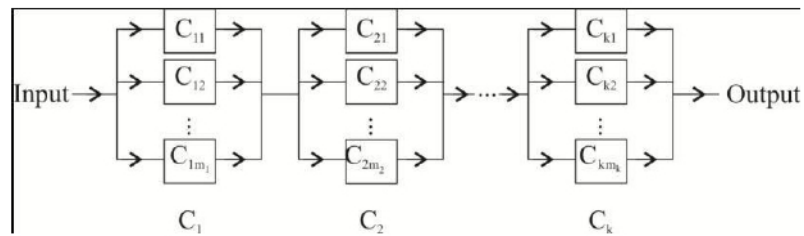


Figure 3: A typical reliability block diagram in terms of k cut sets.

If the i -th cut is named as C_i and the Probability of failure of all the components in C_i is represented by $Pr(C_i)$. We use the addition law of probability, which gives us the probability of occurrence of at least one event from among several events. Thus, if Q_s denotes the unreliability of the system, then

$$Q_s = Pr(C_1 \cup C_2 \cup C_3 \cup \dots \cup C_n) \quad (1)$$

The Reliability of the system is given by

$$R_s = 1 - Q_s \quad (2)$$

2.3 Tie Set Method

The tie set technique is based on the concept of minimal path sets, and it evaluates the system's reliability. This method differs from the cut set method, which evaluates the system's unreliability. The tie set approach begins by identifying the system's minimal path sets.

Let $P_1 P_2 P_3 \dots P_k$ represent the minimal path sets for a system with n components $(1, 2, \dots, n)$. Let m_i denote the number of components in the minimal path set P_i , where $i = 1, 2, \dots, k$.

By definition, all components of a minimal path set must function properly in order for the system to operate correctly along this path. It should be noted that, in terms of reliability, the components of a minimal path set must be connected in series. Having at least one path from input to output $(P_1 P_2 P_3 \dots P_k)$ ensures system success. This indicates that the minimal path sets are linked in parallel from a reliability standpoint. As shown in Fig.2.4, the reliability block diagram of such a system can be represented by P_1, P_2, P_3 and P_k .

In Fig.4, the minimal path sets P_1, P_2, P_3 and P_k are represented by minimal tie sets T_1, T_2, \dots, T_k .

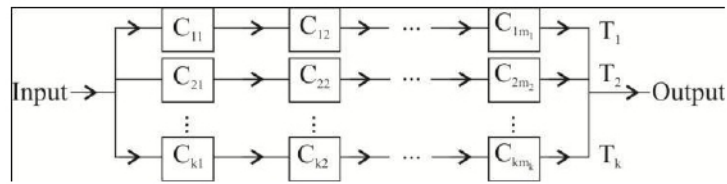


Figure 4: A typical reliability block diagram in terms of k minimal path sets

The concept of union of events applies when assessing the overall system reliability. For system shown in Fig. 3, the reliability is:

$$R_s = Pr(T_1 \cup T_2 \cup T_3 \cup T_4) \quad (3)$$

where $T_i, i = 1, 2, 3, 4$, is the i-th tie set. The above expression can be decomposed into $Pr(T_i)$ which represents the probability that all the components in T_i work. Please note that the definitions of $Pr(C_i)$ and $Pr(T_i)$ are complementary.

The minimal cut/tie sets can be generated by standard algorithms, such as multiplication of the connection matrices, so the reliability evaluation methods based on minimal cut/tie set can be easily implemented on computers [7].

Event Tree

An **event tree** in reliability engineering is a graphical tool used to analyse and model the potential outcomes of an initiating event. It evaluates how different systems or components respond, identifying both successful and failure paths. Event trees are often employed in risk assessments, safety analysis, and system reliability evaluations.

Fault Tree

Fault trees were one method for estimating a subsystem's failure probability. In fact, this technique has been widely utilised for many years to evaluate the reliability of standby, protection, and comprehensive mission-oriented systems, notably safety systems. It is not often utilised for topological systems[2].

Example 2.6 Using the cut set method, evaluate the reliability of the system, which has the reliability block diagram shown in Fig.1 for a mission of 1000 hours. It is given that each component has reliability of 0.95 for a mission of 1000 hours. Assume that the components are independent.

Solution: For applying the cut set method, we first find all minimal cut sets for the system having the reliability block diagram shown in Fig. 2. The minimal cut sets for this system have already been listed in E2. So, we can write

$C_1 = \{1, 3\}, C_2 = \{2, 4\}, C_3 = \{1, 5, 4\}, C_4 = \{3, 5, 2\}$. From a reliability point of view, these cut sets can be shown as in Fig. 3

As mentioned earlier, you should always remember that although C_1, C_2, C_3, C_4 seem

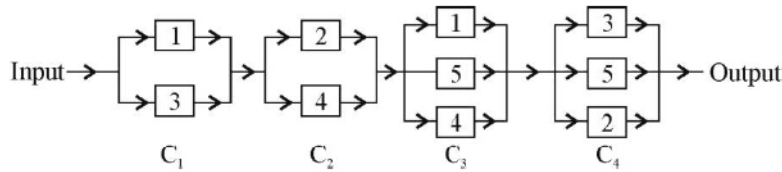


Figure 5: Cut sets for the reliability block diagram of Fig.1

to be connected in series in Fig. 5, we cannot use the concept of series system because these are not made up of independent components. Note that component 4 appears in both C_2 and C_3 , component 2 appears in both C_2 and C_4 , etc.

Therefore, if Q_i denotes the unreliability of the component i , ($i=1,2,3,4,5$) and Q_s that of the system, then by cut set method, we get

$$Q_s = P(C_1 \cup C_2 \cup C_3 \cup C_4)$$

Applying the addition law of probability (inclusion-exclusion rule), we get

$$\begin{aligned} Q_s = & P(C_1) + P(C_2) + P(C_3) + P(C_4) - P(C_1 \cap C_2) - P(C_1 \cap C_3) - P(C_1 \cap C_4) \\ & - P(C_2 \cap C_3) - P(C_2 \cap C_4) - P(C_3 \cap C_4) + P(C_1 \cap C_2 \cap C_3) + P(C_1 \cap C_2 \cap C_4) \\ & + P(C_1 \cap C_3 \cap C_4) + P(C_2 \cap C_3 \cap C_4) - P(C_1 \cap C_2 \cap C_3 \cap C_4) \end{aligned} \quad (4)$$

$$\left(\begin{array}{c} \text{Since } P(A \cup B) = P(A) + P(B) - P(A \cap B) \\ P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \end{array} \right)$$

Let us first evaluate each term of R.H.S. of (1) separately.

$$P(C_1) = Q_1 Q_3 \quad [\because \text{Occurrence of the cutset } C_1 \text{ means that}$$

all the components of C_1 must fail]

$$P(C_2) = Q_2Q_4$$

$$P(C_3) = Q_1Q_5Q_4$$

$$P(C_4) = Q_3Q_5Q_2$$

$$P(C_1 \cap C_2) = P(C_1)P(C_2) = Q_1Q_2Q_3Q_4$$

$$P(C_1 \cap C_3) = P(C_1)P(C_3) = Q_1Q_3Q_4Q_5$$

$$P(C_1 \cap C_4) = P(C_1)P(C_4) = Q_1Q_2Q_3Q_5$$

$$P(C_2 \cap C_3) = P(C_2)P(C_3) = Q_1Q_2Q_4Q_5$$

$$P(C_2 \cap C_4) = P(C_2)P(C_4) = Q_2Q_3Q_4Q_5$$

$$P(C_3 \cap C_4) = P(C_3)P(C_4) = Q_1Q_2Q_3Q_4Q_5$$

$$P(C_1 \cap C_2 \cap C_3) = P(C_1 \cap C_2 \cap C_4) = P(C_1 \cap C_3 \cap C_4) = P(C_2 \cap C_3 \cap C_4) = Q_1Q_2Q_3Q_4Q_5$$

$$P(C_1 \cap C_2 \cap C_3 \cap C_4) = Q_1Q_2Q_3Q_4Q_5$$

We are given that all components (1,2,3,4,5) have the same reliability 0.95 for a mission of 1000 hours. Hence, their unreliability will also be the same.

Let $Q_1 = Q_2 = Q_3 = Q_4 = Q_5 = Q$ (say). Then

$$P(C_1) = Q_1Q_3 = Q^2$$

$$P(C_2) = Q_2Q_4 = Q^2$$

$$P(C_3) = Q_1Q_5Q_4 = Q^3$$

$$P(C_4) = Q_3Q_5Q_2 = Q^3$$

$$P(C_1 \cap C_2) = P(C_1)P(C_2) = Q_1Q_2Q_3Q_4 = Q^4$$

$$P(C_1 \cap C_3) = P(C_1)P(C_3) = Q_1Q_3Q_4Q_5 = Q^4$$

$$P(C_1 \cap C_4) = P(C_1)P(C_4) = Q_1Q_2Q_3Q_5 = Q^4$$

$$P(C_2 \cap C_3) = P(C_2)P(C_3) = Q_1Q_2Q_4Q_5 = Q^4$$

$$P(C_2 \cap C_4) = P(C_2)P(C_4) = Q_2Q_3Q_4Q_5 = Q^4$$

$$P(C_1 \cap C_2 \cap C_3) = P(C_1 \cap C_2 \cap C_4) = P(C_1 \cap C_3 \cap C_4) = P(C_2 \cap C_3 \cap C_4) = Q_1Q_2Q_3Q_4Q_5 = Q^5$$

$$P(C_1 \cap C_2 \cap C_3 \cap C_4) = Q_1Q_2Q_3Q_4Q_5 = Q^5$$

Putting this value in equation (1), we get

$$Q_s = (Q^2 + Q^2 + Q^3 + Q^3) - (5Q^4 + Q^5) + (4Q^5) - Q^5$$

$$Q_s = (2Q^2) + (2Q^3) - (5Q^4) + (2Q^5)$$

Since the reliability of each component for a mission of 1000 hours = 0.95,

The unreliability of each component up to a mission of 1000 hours $Q = 1 - 0.95 = 0.05$

Hence, unreliability of the system (Q_s) is given by

$$Q_s = 2(0.05)^2 + 2(0.05)^3 - 5(0.05)^4 + 2(0.05)^5$$

$$Q_s = 0.005 + 0.00025 - 0.00003125 + 0.000000625$$

$$Q_s = 0.005219375 \quad (5)$$

Therefore the reliability of the system (R_s) is given by

$$R_s = 1 - Q_s = 1 - 0.005219375 = 0.994780625$$

Suppose we call $P(C_1), P(C_2), P(C_3), P(C_4)$ are first order terms;

$P(C_1 \cap C_2), P(C_1 \cap C_3)$ etc.as Second terms;

$P(C_1 \cap C_2 \cap C_3), P(C_1 \cap C_2 \cap C_4)$ etc.as third terms and so on.

Then for components having high reliability (that is having low unreliability), the second and higher order terms can be ignored without affecting the value of reliability much.

In the above example, if we neglect second order and higher order terms, then

$$Q_s = (2Q^2) + (2Q^3) = 0.00525 \text{ and } R_s = 1 - 0.00525 = 0.99475$$

The percentage error in the value of R_s is

$$R_s = \frac{0.994780625 - 0.99475}{0.994780625} \times 100\% = 0.0030786\%$$

i.e., 0.0030% (approx) which can be tolerated. This approximation saves a lot of calculations. This is one of the main advantages of the cut set method

Example 2.7 Evaluate the reliability of the system for which the reliability block diagram is shown in Fig.1 for a mission of 500 hours by using the tie set method. It is given that each component has reliability of 0.95 for a mission of 500 hours. Assume that all components are independent.

Solution: For applying the tie set method, we first find the minimal path sets for the system having the reliability block diagram shown in Fig.1. The minimal path

sets for this system have already been listed in E2. So we repeat them as follows:

$$P_1 = \{1, 2\}, P_2 = \{3, 4\}, P_3 = \{1, 5, 4\}, P_4 = \{3, 5, 2\}$$

Let us denote these minimal path sets by minimal tie sets T_1, T_2, T_3 and T_4 , respectively. The reliability block diagram in terms of tie sets is shown in Fig. 2.6

You should remember one important point here: Although T_1, T_2, T_3 and T_4 seem to

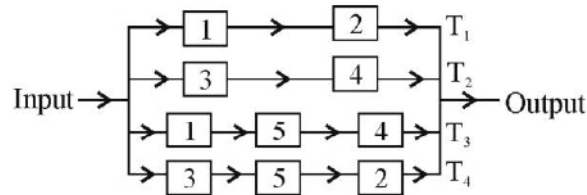


Figure 6: Minimal path sets for Example 2.7

be connected in parallel, you should not use the concept of parallel system because they are not made up of independent components. Note that component 1 appears in both T_1 and T_3 , component 2 appears in T_1 and T_4 , etc.

Therefore, if R_i denotes the reliability of the component i , ($i = 1, 2, 3, 4, 5$), and R_s that of the system, then by tie set method, we have

$$R_s = (T_1 \cup T_2 \cup T_3 \cup T_4)$$

Applying the addition law of probability

$$\begin{aligned} R_s = & P(T_1) + P(T_2) + P(T_3) + P(T_4) - P(T_1 \cap T_2) - P(T_1 \cap T_3) - P(T_1 \cap T_4) \\ & - P(T_2 \cap T_3) - P(T_2 \cap T_4) - P(T_3 \cap T_4) + P(T_1 \cap T_2 \cap T_3) + P(T_1 \cap T_2 \cap T_4) \\ & + P(T_1 \cap T_3 \cap T_4) + P(T_2 \cap T_3 \cap T_4) - P(T_1 \cap T_2 \cap T_3 \cap T_4) \end{aligned} \quad (6)$$

Let us first evaluate each term of R.H.S. of (i) separately

$$P(T_1) = R_1 R_2$$

[Since occurrence Tie set T_1 means that there exist a path between input and output through T_1 This implies that both components 1 and 2 must work successfully]

Similarly,

$$P(T_2) = R_3R_4$$

$$P(T_3) = R_1R_5R_4$$

$$P(T_4) = R_3R_5R_2$$

$$P(T_1 \cap T_2) = P(T_1)P(T_2) = R_1R_2R_3R_4$$

$$P(T_1 \cap T_3) = P(T_1)P(T_3) = R_1R_2R_4R_5$$

*Since occurrence $T_1 \cap T_2$ means means that there exist a path
between input and output through the paths P_1 and P_3 .
This requires that components 1,2,4,5 must work successfully.
But component 1 appears in both paths and hence its probability taken only once.*

$$P(T_1 \cap T_4) = P(T_1)P(T_4) = R_1R_2R_3R_5$$

$$P(T_2 \cap T_3) = P(T_2)P(T_3) = R_1R_3R_4R_5$$

$$P(T_2 \cap T_4) = P(T_2)P(T_4) = R_2R_3R_4R_5$$

$$P(T_3 \cap T_4) = P(T_3)P(T_4) = R_1R_2R_3R_4R_5$$

$$P(T_1 \cap T_2 \cap T_3) = P(T_1)P(T_2)P(T_3) = R_1R_2R_3R_4R_5$$

$$P(T_1 \cap T_2 \cap T_4) = P(T_1)P(T_2)P(T_4) = R_1R_2R_3R_4R_5$$

$$P(T_1 \cap T_3 \cap T_4) = P(T_1)P(T_3)P(T_4) = R_1R_2R_3R_4R_5$$

$$P(T_2 \cap T_3 \cap T_4) = P(T_2)P(T_3)P(T_4) = R_1R_2R_3R_4R_5$$

$$P(T_1 \cap T_2 \cap T_3 \cap T_4) = P(T_1)P(T_2)P(T_3)P(T_4) = R_1R_2R_3R_4R_5$$

Since all components (1,2,3,4,5) have the same reliability 0.95(R, say) for a mission of 500 hours. We can replce R_1, R_2, R_3, R_4, R_5 by R in all expressions obtained above.

So we get

$$P(T_1) = R^2, P(T_2) = R^2, P(T_3) = R^3, P(T_4) = R^3$$

$$P(T_1 \cap T_2) = P(T_1 \cap T_3) = P(T_1 \cap T_4) = P(T_2 \cap T_3) = R^4$$

$$P(T_3 \cap T_4) = R^5$$

$$P(T_1 \cap T_2 \cap T_3) = P(T_1 \cap T_2 \cap T_4) = P(T_1 \cap T_3 \cap T_4) = P(T_2 \cap T_3 \cap T_4) = R^5$$

$$P(T_1 \cap T_2 \cap T_3 \cap T_4) = R^5$$

Putting these values in equation (1), we get

$$R_s = (R^2 + R^2 + R^3 + R^3) - (5R^4 + R^5) + (4R^5) - R^5$$

$$R_s = 2R^2 + 2R^3 - 5R^4 + 2R^5$$

$$R_s = 2(0.95)^2 + 2(0.95)^3 - 5(0.95)^4 + 2(0.95)^5$$

$$R_s = 1.805 + 1.71475 - 4.07253125 + 1.547561875$$

$$R_s = 0.994780655$$

This is the reliability of the given system.

4 Conclusion

The text describes a method for evaluating the reliability of control systems. It uses the concept of cut/tie sets to calculate reliability and can be updated for changes in control objectives or failures. For large systems, it suggests decomposing them into subsystems for easier evaluation. The method is illustrated with an example that demonstrates how proper controller design can improve reliability. Overall, the research indicates that reliability can be a factor in control system design.

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