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Applications of Solutions of Difference Equations in Graph Theory

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Abstract

Graph labeling, which assigns values to the vertices and edges of a graph under specific conditions, has significant applications in real-world problems such as coding theory, radar code design, synch-set codes, missile guidance, and convolution codes with optimal error-correction properties. This study explores the connections between graph labeling and solutions of difference equations by constructing infinite graphs from sequences of real or complex numbers. Each solution of a difference equation induces a labeled graph in the complex plane, where vertex functions extend naturally to edge functions through binary operations over the complex field. Furthermore, the use of complex plane labeling provides a framework for visualizing higher-dimensional relationships in two-dimensional settings, enriching the structural understanding of labeled graphs and their diverse applications.

Key words: Graphs labeling, Difference Equations, Convergent Digraphs, Tensor product, Rotatory Graphs.

AMS classification: 39A06, 05C90, 05C50, 39A10, 05C76.

1 Introduction

Graphs labeling, when the vertices and edges assigned values subject to certain conditions have often been motivated by practical problems. Labeled graphs serves as useful mathematical models for a broad range of applications such as coding theory including the design of good radar type codes, synch-set codes, missile guidance codes and convolution codes with optimal auto correction properties. A systematic presentation of diverse applications of graph labeling is presented in [5]. For a given sequence $\{x_n\}_{n=1}^{\infty}$ of real or complex numbers, it is natural to construct a graph

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G with vertex set $V(G) = \{x_n\}$ and each pair (x_n, x_{n+1}) is connected by an edge e_n, ie $E(G) = \{e_n = x_n x_{n+1}, n = 1, 2, \ldots\}$. If a graph G contains either infinitely many vertices or infinitely many edges, then the graph G is called infinite Graph [9]. Solution of a difference equation is a sequence of real of complex numbers satisfying the given difference equation (see [1],[15],[16],[16]). Hence each solution of difference equation induces a graph in the complex plane. Given a graph G=(V,E), the set $\mathbb{C}$ of complex numbers and a commutative binary operation $* : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, every vertex function $f : V(G) \to \mathbb{C}$ induces an edge function $g_f : E(G) \to \mathbb{C}$ such that $g_f(u, v) = f(u) * f(v)$ for all $uv \in E(G)$(see [22]). Since every 3 - D diagram is displayed in a two dimensional screen, we take the label set as the subset of two dimensional complex plane.

2 Preliminaries of Tensor Product

Definition 2.1 In graph theory, the ***tensor product*** $G \times H$ of graphs G and H is a graph such that

1.The vertex set of $G \times H$ is the cartesian product $V(G) \times V(H)$.

2.Any two vertices (u, u') and (v, v') are adjacent in $G \times H$ iff u' is adjacent with v' and v is adjacent with v.

The tensor product is also called the ***direct product, categorical product, cardinal product, relational product, kronecker product, weak direct product (or) conjection.*** As on operation on binary relations, the tensor product was introduced by Alfred north whitehead and Bertrand Russell in their principia Mathematica(1912). It is also equivalent to the Kronecker product of the adjacency matrices of the graphs (Weichsel 1962).

The notation $G \times H$ is also sometimes used to refer to the cartesion product of graphs, but more commonly refers to the tensor product. The cross symbol shows visually the two edges resuliting from the tensor product of two edges.

3 Properties of tensor product

The tensor product is the category-theoretic product in the category of graphs and graph homomorphisms. That is, there is a homomorphism from $G \times H$ to G and to H (given by projection onto each coordinate of the vertices) such that any other graph that has a homomorphism to both G and H has a homomorphism to $G \times H$ that factors through the homomorphisms to G and H.

The adjacency matrix of $G \times H$ is the tensor product of the adjacency matrices of G and H.

If a graph can be represented as a tensor product, then there may be multiple different representations (tensor products do not satisfy unique factorization) but each representation has the same number of irreducible factors. Imrich(1998) gives a polynomial time algorithm for recognizing tensor product graphs and finding a factorization of any such graph.

If either G (or)H is bipartite, then so is their tensor product $G \times H$ is connected if and only if both factors are connected and atleast one factor is nonbipartite. The tensor product $K_2 \times G$ is sometimes called the **double cover** of G;if G is already bipartite, its double cover is the disjoint union of two copies of G.

The Hedetniemi conjecture gives a formula for the chromotic number of a tensor product.

4 Convergent Digraphs

Definition 4.1 Let G_0 be an infinite digraph whose vertex set $V(G_o) \subset L$. A finite set $\{v_{oi}\}_{i=1}^{\ell} \subset L$ is said to be a ***limiting set of*** $V(G_o)$ if for any $\epsilon > 0$ there exist a vertex $v_{\epsilon i} \in N_\epsilon(v_{oi}) \bigcap V(G_o)$ and a directed path starting from each vertex $v_x \in \left\{V(G_o) - \bigcup_{i=1}^{\ell} N_\varepsilon(v_{oi})\right\}$ to $v_{\epsilon i}$ for $i = 1, 2, 3, ...\ell$. Then $G = G_o \cup \{v_{oi}\}_{i=1}^{\ell}$ is called a ***convergent digraph***.

Definition 4.2 If $V(G_i) \subseteq V(G_j)$ and $E(G_i) \subseteq E(G_j)$, then we say that $G_i \subseteq G_j$. Let $G_1 \subseteq G_2 \subseteq ...G_n \subseteq G_{n+1} \subseteq ....$ be an increasing sequence of digraphs. The smallest convergent digraph $\overline{G} \supseteq \bigcup G_i$ is called ***maximal limiting digraph***. If $g_1 \supseteq g_2 \supseteq ...g_n \supseteq g_{n+1} \supseteq ....$ is a decreasing sequence of digraphs, then the largest convergent digraph $\underline{g} \subseteq \bigcap G_i$ is called ***minimal limitting digraph***.

Theorem 4.3 If G is an infinite convergent digraph with limiting set $\{v_{oi}\}_{i=1}^{\ell} \subset L$, then there exists an increasing sequence of digraphs $\{G_n\}_{n=1}^{\infty}$ whose maximal limiting digraph is G and a decreasing sequence of digraphs $\{g_n\}_{n=1}^{\infty}$ whose minimal limiting digraph is K_ℓ^c.

Proof: Let $N_{\frac{1}{n}}(v_{oi}) \subset L$ be the $\frac{1}{n}$-neighbourhood of the vertex v_{oi}.

Define $G_n = G - \left[V(G) \bigcap \left\{\bigcup_{i=1}^{\ell} N_{1/n}(v_{oi})\right\}\right]$ for $n = 1, 2, 3, ...$ and $g_n = G - \left[V(G) \bigcap \left\{\bigcup_{i=1}^{\ell} N_{1/n}(v_{oi})\right\}^c\right]$ where the complementation is the usual.

Then $\{G_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are respectively increasing and decreasing sequences of digraphs. Since $G_n \to G - \{v_{oi}\}_{i=1}^{\ell}$ and $g_n \to \{v_{oi}\}_{i=1}^{\ell}$ as $n \to \infty$, the proof follows from the definitions 4.1 and 4.2.

Theorem 4.4 For a given finite set $\{v_{oi}\}_{i=1}^{\ell} \subset R \cup C$ there exists a convergent digraph G with an increasing sequence of digraphs $\{G_n\}_{n=1}^{\infty}$ and a decreasing sequence of digraphs $\{g_n\}_{n=1}^{\infty}$ such that the maximal limiting digraph of $\{G_n\}_{n=1}^{\infty}$ is G and the minimal limiting digraph of $\{g_n\}_{n=1}^{\infty}$ is K_{ℓ}^{c}.
Proof: The digraph G with $V(G) = \bigcup_{i=1}^{\ell} \left\{ \{v_{oi}\} \bigcup_{r=1}^{\infty} \left\{ v_{oi} + \frac{1}{r} \right\} \right\}$ and
$E(G) = \bigcup_{i=1}^{\ell} \left[\left\{ v_{oi} \to v_{o(i+1)} \right\} \cup \left\{ \left(v_{oi} + \frac{1}{r}\right) \rightleftarrows \left(v_{oi} + \frac{1}{(r+1)} \right) \right\}_{r=1}^{\infty} \right]$ satisfies the conditions of Theorem 4.4 and hence the proof follows.

5 Rotatory Graphs of Difference Equations

Definition 5.1 If ℓ and n are positive integers, then the equation

$$f(k, v(k), v(k+\ell), ..., v(k+n\ell)) \;=\; 0 \tag{1}$$

is called a ***generalised difference equation.*** Let $\{v(k)\}_{k=1}^{\infty}$ be a solution of the eqn.(1). The digraph G with $V(G) = \{v(k)\}_{k=1}^{\infty}$ and $E(G) = \{v(k) \to v(k+1)\}_{k=1}^{\infty}$ is called ***solution graph*** of equation (1). Finite union of solution graphs of equation (1) is called ***difference equational digraph.***

Definition 5.2 Let $V(G_R) = \{v(k)\}_{k=1}^{\ell} \subset L$, $v(\ell+1) = v(1)$ and
$E_{rot}(G_R) = \{v(k) \to v(k+1)\}_{k=1}^{\ell}$. The digraph $G_R \;=\; [V(G_R), nE_{rot}(G_R)]$, having 'n' times $E_{rot}(G_R)$ as edge set is called ***n-rotatory block*** of length ℓ and n (n may be ∞) is called ***multiplicity*** of the rotatory block. Union of finite number of rotatory blocks, union of finite number of rotatory blocks of same length and union of finite number of finite-rotatory blocks of same length as well as same multiplicity are called ***rotatory, uniform rotatory*** and ***regular rotatory graphs*** respectively.

Lemma 5.3 For a given uniform rotatory digraph G of length 2ℓ, there exists finite number of distinct sequences of complex numbers, $\{v_r(k)\}_{k=1}^{\infty}$ for $r = 1, 2, ...n$ associated with the labels of vertices of G and satisfying the conditions
$\frac{v_r(k)}{v_r(k+\ell)}$ = -1 for $k = 1, 2, 3, ...\infty$ and $r = 1, 2, 3, ...n$.

Proof: By defnition 5.2, $G = \bigcup_{r=1}^{n} [V(G_{R_r}), m_r E_{rot}(G_{R_r})]$. The proof follows by taking $v_r(k) = re^{\frac{i\pi k}{\ell}}$ and $V(G_{R_r})= \{v_r(k)\}_{k=1}^{2\ell}$.

Definition 5.4 Let $G_s = [V(G_s), E(G_s)]$ be a digraph. If there exists a positive integer m and a sequence $\{v(k)\} \subset C$ associated with the labels of $V(G_s)$ such that $\frac{v(k)}{v(k+m)} < 0$ for all k and $E(G_s)$ is of the form $\{v(k) \to v(k+1)\}$, then the digraph G_s is called ***spiral branch*** of length $2m$. The spiral branch of length 2 is called ***oscillatory branch***. Union of finite number of spiral branches is called ***spiral graph***.

6 Conclusion

The solutions of difference equations provide a powerful bridge between discrete dynamical systems and graph theory. By representing solutions as sequences and mapping them onto vertices and edges, one can construct infinite or finite graphs that capture the structural and functional properties of the underlying equations. This correspondence not only enriches the study of graph theory but also facilitates applications in areas such as network modeling, coding theory, and computational mathematics, where the dynamics of sequences naturally translate into graph-based representations.

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