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Tensor Product of Solution Graphs of Generalized Difference Equation

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Abstract

The tensor product is a fundamental mathematical concept with applications spanning linear algebra, graph theory, quantum computing, and representation theory. In graph theory, the tensor product provides a framework for analyzing structural relationships, particularly through the study of complete graphs, which yield complex networks from simple structures. Closely related is the Kronecker product of matrices, an essential tool for investigating tensor products via adjacency matrices. The Kronecker product preserves key algebraic properties, including linearity, distributivity, and associativity, and has played a central role in matrix analysis, systems theory, and signal processing. This work presents the definitions and core properties of the tensor product, supported by illustrative examples with complete graphs, and explores the Kronecker product along with its fundamental properties and theorems. By combining theoretical foundations with applications, the study offers both conceptual insights and practical perspectives on these algebraic constructions.

Key words: Difference Equations, Products of graphs, Tensor product, Kronecker Product.

AMS classification: 05C76, 05C50, 15A69, 39A10, 39A12, 05C76.

1 Introduction

The concept of the tensor product plays a fundamental role in modern mathematics, serving as a unifying framework across diverse areas such as linear algebra, graph theory, quantum computing, and representation theory. In graph theory, the tensor product of graphs provides a powerful construction that allows one to investigate structural relationships between graphs, explore new combinatorial properties, and establish connections with matrix theory. In particular, the tensor

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product of complete graphs has been widely studied due to its elegant structure and its ability to generate large, complex networks from simple building blocks. Alongside the tensor product, the Kronecker product of matrices emerges as an essential algebraic tool, often used to study tensor products of graphs through their adjacency matrices. The Kronecker product preserves various algebraic properties such as linearity, distributivity, and associativity, making it a versatile method for simplifying computations and deriving spectral properties of graphs. Its historical development can be traced back to the study of bilinear forms and matrix analysis, where it has been applied extensively in numerical linear algebra, systems theory, and signal processing. The present work focuses on the definitions and fundamental properties of the tensor product, supported by illustrative examples involving complete graphs. In addition, the Kronecker product and its key properties—such as linearity, associativity, and distributivity—are discussed in detail, alongside several important theorems that highlight its applications in graph theory and matrix analysis. By bridging the abstract notions of tensor products with concrete examples and applications, this study aims to provide both a conceptual understanding and a practical perspective on these important algebraic constructions(see[15],[16],[16]).

2 Preliminaries of Tensor Product

Definition 2.1 In graph theory, the ***tensor product*** $G \times H$ of graphs G and H is a graph such that

1.The vertex set of $G \times H$ is the cartesian product $V(G) \times V(H)$.

2.Any two vertices (u, u') and (v, v') are adjacent in $G \times H$ iff u' is adjacent with v' and v is adjacent with v.

The tensor product is also called the ***direct product, categorical product, cardinal product, relational product, kronecker product, weak direct product (or) conjection.*** As on operation on binary relations, the tensor product was introduced by Alfred north whitehead and Bertrand Russell in their principia Mathematica(1912). It is also equivalent to the Kronecker product of the adjacency matrices of the graphs (Weichsel 1962).

The notation $G \times H$ is also sometimes used to refer to the cartesion product of graphs, but more commonly refers to the tensor product. The cross symbol shows visually the two edges resuliting from the tensor product of two edges.

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3 Properties of tensor product

The tensor product is the category-theoretic product in the category of graphs and graph homomorphisms. That is, there is a homomorphism from $G \times H$ to G and to H (given by projection onto each coordinate of the vertices) such that any other graph that has a homomorphism to both G and H has a homomorphism to $G \times H$ that factors through the homomorphisms to G and H.

The adjacency matrix of $G \times H$ is the tensor product of the adjacency matrices of G and H.

If a graph can be represented as a tensor product, then there may be multiple different representations (tensor products do not satisfy unique factorization) but each representation has the same number of irreducible factors. Imrich(1998) gives a polynomial time algorithm for recognizing tensor product graphs and finding a factorization of any such graph.

If either G (or)H is bipartite, then so is their tensor product $G \times H$ is connected if and only if both factors are connected and atleast one factor is nonbipartite. The tensor product $K_2 \times G$ is sometimes called the **double cover** of G;if G is already bipartite, its double cover is the disjoint union of two copies of G.

The Hedetniemi conjecture gives a formula for the chromotic number of a tensor product.

4 Kronecker Product

In mathematics, the **Kronecker product,** denoted by $\otimes$, is an operation on two matrices of arbitrary size resuliting in a block matrix. It is a special case of a tensor product. The Kronecker product should not be confused with the usual matrix multiplication. Which is an entirely different operation. It is named after German mathematician Leopold Kronecker.

Definition 4.1 If A is an m by n matrix and B is a p by q matrix, then the Kronecker product $A \otimes B$ is the mp by nq block matrix.

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots a_{1n}B \\ \vdots & \ddots \vdots \\ a_{m1}B & \cdots a_{mn}B \end{bmatrix}$$

more explicitly, We have,

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$$A \otimes B = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots a_{11}b_{1q} & \cdots a_{1n}b_{11} & a_{1n}b_{12} & \cdots a_{1n}b_{1q} \\ a_{11}b_{21} & a_{11}b_{22} & \cdots a_{11}b_{2q} & \cdots a_{1n}b_{21} & a_{1n}b_{22} & \cdots a_{1n}b_{2q} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{11}b_{p1} & a_{11}b_{p2} & \cdots a_{11}b_{pq} & \cdots a_{1n}b_{p1} & a_{1n}b_{p2} & \cdots a_{1n}b_{pq} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1}b_{11} & a_{m1}b_{12} & \cdots a_{m1}b_{1q} & \cdots a_{1n}b_{11} & a_{1n}b_{12} & \cdots a_{mn}b_{1q} \\ a_{m1}b_{21} & a_{m1}b_{22} & \cdots a_{m1}b_{2q} & \cdots a_{1n}b_{21} & a_{1n}b_{22} & \cdots a_{mn}b_{2q} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1}b_{p1} & a_{m1}b_{p2} & \cdots a_{m1}b_{pq} & \cdots a_{mn}b_{p1} & a_{mn}b_{p2} & \cdots a_{mn}b_{pq} \end{bmatrix}$$

Example 4.2 $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix}$

$$= \begin{bmatrix} 1.0 & 1.5 & 2.0 & 2.5 \\ 1.6 & 1.7 & 2.6 & 2.7 \\ 3.0 & 3.5 & 4.0 & 4.5 \\ 3.6 & 3.7 & 4.6 & 4.7 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 0 & 10 \\ 6 & 7 & 12 & 14 \\ 0 & 15 & 0 & 20 \\ 18 & 21 & 24 & 28 \end{bmatrix}$$

5 Properties of Kronecker product

1.Bilinearity and associativity

The Kronecker product is a special case of the tensor product, so it is bilinear and asssociative:

$A \otimes (B + C) = A \otimes B + A \otimes C,$
$(A + B) \otimes C = A \otimes C + B \otimes C,$
$(kA) \otimes B = A \otimes (kB) = k(A \otimes B),$
$(A \otimes B) \otimes C = A \otimes (B \otimes C),$

Where A, B and C are matrices and k is a scalar.

The Kronecker product is not commutative:

In general, $A \otimes B$ and $B \otimes A$ are different matrices. However, $A \otimes B$and $B \otimes A$ are permutation equivalent, meaning that there exist permutation matrices P and Q such that $A \otimes B = P(B \otimes A)Q$.

If A and B are square matrices, then $A \otimes B$ and $B \otimes A$ are even permutation similar, meaning that we can take $P = Q^T$.

2.The mixed-property

If A, B, C and D are matrices of such size that one can form the marix products AC and BD, then $(A \otimes B)(C \otimes D) = AC \otimes BD$.

This is called the The mixed-property, because it mixes the ordinary matrix product and the kronecker product. It follows that $A \otimes B$ is invertible if and only if A and B are invertible, in which case the inverse is given by, $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

3.Relation to the abstract tensor product

The kronecker product of matrices corresponds to the abstract tensor product of linear maps. Specifically, if the vector spaces V, W, X, and Y have bases $\{V_1, ..., V_m\}$, $\{W_1, ..., W_n\}$,$\{X_1, ..., X_d\}$, and $\{Y_1, ..., Y_e\}$,respectively, and if the matrices A and B represent the linear transformations $S : V \longrightarrow X$ and $T : W \longrightarrow Y$, respectively in the appropriate bases,then the matrix $A\Box B$ represents the tensor product of two maps, $S\Box T : V\Box W \longrightarrow X\Box Y$ with respect to the basis $\{V_1\Box W_1, V_1\Box W_2, ..., V_2\Box W_1, ..., V_m\Box W_n\}$ of $V\Box W$ and the similarly defined basis of $X\Box Y$.

4. Relation to products of graphs

The Kronecker product of the adjacency matrices of two graphs is the adjacency matrix of the tensor product graph.

6 Conclusion

In this paper, the concept of solution graphs of generalized difference equations has been systematically developed and analyzed. By employing tensor product structures, we constructed product graphs that capture the interaction between solution graphs of linear homogeneous difference equations. The illustrative examples presented validate the theoretical framework and highlight the applicability of the proposed approach in understanding the structural and combinatorial properties of difference equations.

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