2456-8686, ix(ii)2025:029-038
https://doi.org/10.26524/cm216

ORIGINAL RESEARCH **Open Access**

Exploring Transcritical and Pitchfork Bifurcations in Parameter-Varying Systems

Geethalakshmi S[1], Kalaiyarasi S[2], Saraswathi D[3] and Vignesh K[4]

Received: 20 August 2025/ Accepted:10 October 2025 / Published online: 20 December 2025

Abstract

This work explores with an emphasis on limited trajectories close to the origin, as well as their pullback attraction and asymptotic instability, this paper examines non-autonomous trans critical bifurcations as well as pitchfork bifurcation. It examines the circumstances in which bifurcation takes place, using the dynamics outlined in system.

Key Words: Bifurcation , transcritical bifurcation, pitchfork bifurcation, Non-autonomous , Unstable, Pullback Trajectory.

AMS Classification: 34C23, 34C37, 34D05, 34A34.

1 Introduction

1.1 Definitions

In a ‘transcritical bifurcation’ a non-autonomous system that the non-zero trajectory is in some sense ‘localised’ near the origin, and that the system in the past (pullback attraction and asymptotic instability).

[1] Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur, Tamil Nadu, South India. Email:geethathiru126@gmail.com
[2] Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur, Tamil Nadu, South India. Email: skalaiarasi2@gmail.com
[3] Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur, Tamil Nadu, South India. Email: dswathisaranya@gmail.com
[4] Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur, Tamil Nadu, South India. Email: vigneshgokul210@gmail.com

The system $\dot{x} = f(x,t,\lambda)$ undergoes a local transcritical bifurcation at $x = 0$, $\lambda = 0$ if there exists a $\lambda_0 > 0$ and an $\epsilon > 0$ such that,

(i) for all $-\lambda_0 < \lambda < 0$ the zero solution is locally pullback attracting within $(-\epsilon, 0]$ and pullback attracting within $[0, \epsilon)$; and there is another negative complete trajectory $x_\lambda(t)$ within $(-\epsilon, 0)$ that is asymptotically unstable and satisfies,

$$x_\lambda(t) \to 0 \text{ as } \lambda \to 0 \quad (1.1.1)$$

(ii) for$\lambda = 0$ the zero solution is asymptotically unstable but still pullback attracting within $[0, \epsilon)$; and

(iii) for$0 < \lambda < \lambda_0$ the zero solution is asymptotically unstable, and there is another positive complete trajectory $x_\lambda(t)$ within $(0, \epsilon)$ the satisfies

$$x_\lambda(t) \to 0 \; as \; \lambda \to 0 \quad (1.1.2)$$

and is pullback attracting within $(0, \epsilon)$

2 Conditions For Localised Bifurcating Solutions

2.1 Lemma

Suppose that , $\lim_{n\to\infty} inf\, g(t) > 0$ (2.1.1)

And $0 < m = \lim_{t\to-\infty} inf \frac{f(t)}{g(t)} \leq \lim_{t\to-\infty} sup \frac{f(t)}{g(t)} = M < +\infty$ (2.1.2)

Then for $\lambda > 0$

$$\lambda m \leq \lim_{t\to-\infty} inf\, x_\lambda(t) \leq \lim_{t\to-\infty} sup x_\lambda(t) \leq \lambda M, \text{for } \lambda < 0, \quad (2.1.3)$$

We have $\lim_{s\to-\infty} inf \frac{e^{\lambda F(s)}}{\int_s^t e^{\lambda F(r)} g(r) dr} > -m\lambda$ (2.1.4)

where F is an anti-derivative f.

Proof

For any $K > M$, there exists a T such that for all $t \leq T$ we have $g(t) > 0$ and

$\frac{f(t)}{g(t)} \leq k$, for t

$$\int_{-\infty}^{t} e^{\lambda F(s)} g(s) ds \geq \frac{1}{k} \int_{-\infty}^{t} e^{\lambda F(s)} f(s) ds$$

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$$\geq \frac{1}{k}\left[\frac{e^{\lambda F(s)}}{\lambda}\right]_{s=-\infty}^{t}$$

$$\geq \frac{1}{\lambda k} e^{\lambda F(t)}$$

Since, $F(t) \to \infty as t \to -\infty$ by (2.1.1) & (2.1.2)

$$\therefore x_\lambda(t) = \frac{e^{\lambda F(t)}}{\int_{-\infty}^{t} e^{\lambda F(s)} g(s) ds} \leq k\lambda \text{ for all } t \leq T$$

And lim $_{t\to-\infty}$ sup $x_\lambda(t)$= M λ

For lower bound the proof is similar as upper bound for any $k < m$ then there exists T such that ,

$$\frac{f(t)}{g(t)} > k \text{for all } t \leq T$$

3 The General Case

We now consider $\dot{x} = G(t, x, \lambda)$, where the R.H.S has Taylor expansion,

$$G(t,x,\lambda) = G + G_x x + G_\lambda \lambda + \frac{1}{2} G_{xx} x^2 + G_{x\lambda} x\lambda + \frac{1}{2} G_{\lambda\lambda} \lambda^2 + \frac{1}{6} G_{xxx} x^3 + \frac{1}{3} G_{xx\lambda} x^2 \lambda$$

$$+ \frac{1}{3} G_{x\lambda\lambda} x\lambda^2 + \frac{1}{6} G_{\lambda\lambda\lambda} \lambda^3 + \cdots$$

(all expressions involving G and its derivatives on the R.H.S are evaluated at $(t, 0, 0)$).

We assume that $G(t, 0, \lambda) = 0$ for all t and λ, and that $G_x(t, 0, 0) = 0$.

$$\frac{\partial^k G}{\partial^k \lambda(t,0,\lambda)} = 0, \text{ for all t and } k \epsilon Z_+$$

$$\therefore G(t,x,\lambda) = \lambda[G_{x\lambda} + \frac{1}{3} G_{x\lambda\lambda} \lambda + \cdots] x + [\frac{1}{2} G_{xx} + \frac{1}{6} G_{xxx} x + \frac{1}{3} G_{xx\lambda} \lambda + \cdots] x^2$$

3.1 Theorem

Consider, $\dot{x} = G(t, x, \lambda)$ and assume that,

$G(t, 0, \lambda) = 0$ for all $\lambda \epsilon \mathbb{R}$ and $G_x(t, 0,0) = 0$

Set $f(t) = G_{x\lambda}(t, 0, 0)$ and $g(t) = -\frac{1}{2} G_{xx}(t, 0,0)$

And rewrite the equation as

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$$\dot{x} = \lambda[f(t) + \lambda\phi(t,\lambda)]x - [g(t) + \gamma(t,x,\lambda)]x^2$$

Where,

$$\phi(t,0) = \frac{1}{3}G_{x\lambda\lambda}(t,0,0) \quad \text{and} \quad \gamma(t,0,0) = 0 \qquad (3.1.1)$$

Assume that $\lim_{t\to\pm\infty} g(t) > 0$ (3.1.2)

That

$$0 < m = \lim_{t\to\pm\infty} inf \frac{f(t)}{g(t)} \leq \lim_{t\to\pm\infty} sup \frac{f(t)}{g(t)} = M < +\infty \qquad (3.1.3)$$

and that $|\phi(t,\lambda)| \leq h(t), \ |\Upsilon_\lambda(t,x,\lambda)| \leq h(t), and$

$$|\gamma_x(t,x,\lambda)| \leq h(t) \qquad (3.1.4)$$

where $\lim_{t\to\pm\infty} sup \frac{h(t)}{g(t)} \leq K$

Then there is a local transcritical bifurcation as λ passes through zero.

(i) When $\lambda < 0$ the 'unstable' trajectory is pullback repelling in $(-\epsilon, 0)$;

(ii) When $\lambda = 0$ the origin is locally forwards attracting in $\mathbb{R}_+$; and

(iii) When $\lambda > 0$ the pullback attracting trajectory $x_\lambda(\cdot)$ is forwards attracting in $(0,\epsilon)$.

For a transcritical bifurcation in the autonomous equation,

$$\dot{x} = f(x,\lambda)$$

$$f(0,\lambda) = 0, \qquad f_x(0,0) = 0, f_{x\lambda}(0,0) > 0 \ and \ f_{xx}(0,0) < 0$$

$$if \ G(t,x,\lambda) = f(x,\lambda)$$

Proof

We assume that $|\lambda| \leq \epsilon$, where ϵ will be chosen 'sufficiently small'.

From (3.1.1) and (3.1.4)

$$|\gamma(t,x,\lambda)| \leq h(t)[|x| + |\lambda|] \qquad (3.1.5)$$

The origin is locally pullback attracting in $(-\epsilon, \epsilon)$ for $\lambda < 0$

$0 < x(t,s,x_s) \leq \epsilon$ We have,

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$0 \leq x(t,s,x_s) \leq v(t,s,x_s)$ where $v(t)$ solves

$\dot{v} = \lambda[(f(t) + \epsilon h(t)]v - [g(t) - 2\epsilon h(t)]v^2$ with $v(s) = x_s$

there exists T such that if $s \leq t \leq T$ changing T

$$\Rightarrow |h(t)| \leq \frac{K^{\mathsf{I}} f(t)}{m} (\text{for some } K^{\mathsf{I}} > K)$$

to deduce that

$$\dot{v} \leq \lambda(1 - (\epsilon K'/m)f(t)v \quad \Rightarrow v(t) \leq e^{(1-\epsilon K'/m)\lambda(F(t)-F(t_0))v(t_0)}$$

decreasing T, so that $f(t) > 0$ for all $t \leq T$,

for $s \leq t \leq T$ we have $v(t,s,x_s) \leq \epsilon$ provided that $0 < x_s \leq \epsilon$ and hence,

$x(t,s,x_s) \leq v(t,s,x_s)$ remains valid,

$\lim_{s\to-\infty} S(t,s)x_s = 0$ for all $t \leq T$

Since $S(T\ ,t)$ is continuous and zero is invariant we have,

$$\lim_{s\to-\infty} S(T,s)x_s = S(T,t)[\lim_{s\to-\infty} S(t,s)x_s]$$

$$= S(T,t)0$$

$$= 0, \quad \text{for all } T\epsilon\mathbb{R}$$

and the origin is pullback attracting within [0,Є).

While $-\epsilon \leq x(t,s,x_s) \leq 0$ we have,

$u(t,s,x_s) \leq x(t,s,x_s) \leq 0$ where $u(t)$ solves

$$\dot{u} = \lambda[f(t) - \epsilon h(t)]u - [g(t) + 2\epsilon h(t)]u^2 \; with \; u(s) = x_s$$

$$\therefore u \geq -\epsilon$$

$$\dot{u} \geq \lambda f(t)[(1 - \epsilon K/m)u] - (1 + 2\epsilon K)u^2/m$$

For T chosen such that $f(t) > 0$ for all $t \leq T$, and

for $\quad 0 \geq x_s \geq -\lambda(m - \epsilon K)/m(1 + 2\epsilon K)$

$$\lim_{s\to-\infty} S(t,s)x_s = 0 \;\; forall \; t \leq T,$$

the origin is locally pullback attracting within (−Є,0].

When λ=0, $|x|$≤€ we have

$$\dot{x} \le -[g(t) - 2\epsilon h(t)]x^2$$

The pullback attraction of zero solution within [0,€) asymptotic instability of zero.

Since for $x_s < 0$ we have,

$$x(t, s, x_s) \to 0 \; as \; t \to -\infty.$$

There is a positive trajectory that is pullback attracting in $[0, \infty)$ when $\lambda > 0$

$|x(t, s, x_s)|,\ |\lambda| < \epsilon$we have,

$$u(t, s, x_s) \le x(t, s, x_s) \le v(t, s, x_s) \tag{3.1.6}$$

where$u(t, s, x_s)$ and $v(t, s, x_s)$ are the solutions of,

$$\dot{u} = \lambda \underbrace{[f(t) - \epsilon h(t)]}_{f_-(t)} u - \underbrace{[g(t) + 2\epsilon h(t)]}_{g_+(t)} u^2 \; with \; u(s) = x_s$$

And

$$\dot{v} = \lambda \underbrace{[f(t) + \epsilon h(t)]}_{f_+(t)} v - \underbrace{[g(t) - 2\epsilon h(t)]}_{g_-(t)} v^2 \;\; with \; v(s) = x_s \tag{3.1.7}$$

We have an explicit form for the solution of (3.1.7)

$$v(t) = \frac{e^{\lambda F_+(t)}}{{x_s}^{-1} e^{\lambda F_+(s)} + \int_s^t e^{\lambda F_+(r)} g_-(r) dr}$$

Using "$0 < m = \lim_{t \to -\infty} inf \frac{f(t)}{g(t)} \le \lim_{t \to -\infty} sup \frac{f(t)}{g(t)} = M < +\infty$

For λ and x_s sufficiently small, $v(t) \le \epsilon$ for all t≤0.

(3.1.6) remains valid for all such t.

Due to the two-sided balance and between h and g we define upper and lower solutions,

$$x_+(t) = \frac{e^{\lambda F_+(t)}}{\int_{-\infty}^{t} e^{\lambda F_+(r)} g_-(r) dr}$$

$$x_-(t) = \frac{e^{\lambda F_-(t)}}{\int_{-\infty}^{t} e^{\lambda F_-(r)} g_+(r) dr}$$

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The pullback attractors of upper and lower equations.

Using $x_{-}(t) \le \lim_{s\to-\infty} inf\, x(t,s,x_s) \le \lim_{s\to-\infty} sup\, x(t,s,x_s) \le x_{+}(t)$

There exists a pullback attractor A(t) of interval (0,Є).

There are two solutions x_1(t) and x_2(t) such that

A(t) = $[x_1(t).x_2(t)]$. and We have,

$x_{-}(t) \le x_j(t) \le x_{+}(t)$ for j=1,2

Consider, $z(t) = x_1(t) - x_2(t)$ then

$$\frac{dz}{dt} \le \lambda[f(t) + Єh(t)]z - g(t)(x_1 + x_2)z - [\Upsilon(t, x_1, \lambda){x_1}^2 - \Upsilon(t, x_2, \lambda){x_2}^2]$$

$$\le \lambda f_{+}(t)z - g(t)(x_1 + x_2)z - \Upsilon(t, x_1, \lambda)({x_1}^2 - {x_2}^2) + [\Upsilon(t, x_1, \lambda) - \Upsilon(t, x_2, \lambda)]{x_2}^2$$

$$\le \lambda f_{+}(t)z - g(t)(x_1 + x_2)z + 2Є\, h(t)[x_1 + x_2]z + Єx_1 h(t)z$$

$$\le [\lambda f_{+}(t) - (2g(t) - 5Єh(t))x_{-}(t)]z$$

Since $2g(t) - 5Єh(t) \ge \frac{2-5\epsilon k}{1+Єk} g_{+}(t)$

$$\frac{dz}{dt} \le \lambda\left[f(t) - \frac{2-5\epsilon k}{1+2Єk}\frac{g_{+}(t)e^{\lambda F_{-}(t)}}{\int_{-\infty}^{t} e^{\lambda F_{-}(r)} g+(r)dr}\right]z$$

$$Z(t) \le Z(t_o)e^{I(t,t_0)}$$

Where $I(t, t_0) = \int_{t_0}^{t} \lambda f_{+}(s) - \frac{2-5\epsilon k}{1+2Єk}\frac{g_{+}(s)e^{\lambda F_{-}(t)}}{\int_{-\infty}^{t} e^{\lambda F_{-}(r)} g+(r)dr} ds$

$$= \lambda(F_{+}(t) - F_{+}(t_0)) - \frac{2-5\epsilon k}{1+2Єk}\left[In \int_{-\infty}^{s} e^{\lambda F_{-}(s)} g_{+}(r)\right]_{s=t_0}^{t}$$

Now $\frac{f-}{M} \le g_{+} \le \frac{1+2Єk}{m-Єk} f_{-}$

And $\left[In \int_{-\infty}^{s} e^{\lambda F_{-}(r)} g+(r)dr\right]_{s=t_0}^{t} \ge In(\frac{1}{\lambda M} e^{\lambda F_{-}(t)}) - In(\frac{1+2Єk}{\lambda(m-Єk)} e^{\lambda F_{-}(t_0)})$

so $= \lambda(F_{-}(t) - F_{-}(t_0) + In \frac{m-Єk}{M(1+2Єk)}$

$$\therefore 1 \le \lambda(F_{+}(t) - F_{+}(t_0) - \frac{2-5ЄK}{1+2Єk}\lambda(F_{-}(t) - F_{-}(t_0) + c_Є$$

Where $c_{\epsilon} = -\frac{2-5\epsilon K}{1+2\epsilon k} In \frac{m-\epsilon k}{M(1+2\epsilon k)} > 0$

Since $f_{-} \geq \frac{1-(\epsilon k/m)}{1+(\epsilon k/m)} f_{+}$

we also have, $F_{-}(t) - F_{-}(t_0) \geq \frac{1-(\epsilon k/m)}{1+(\epsilon k/m)} [F_{+}(t) - F_{+}(t_0)]$

and $T \leq \lambda(F_{+}(t) - F_{+}(t_0)) \left[1 - 2\frac{\left(1-\frac{5}{2}\epsilon k\right)(1-\epsilon k/m)}{(1+2\epsilon k)(1+\epsilon k/m}\right] + C_{\epsilon}$

for Є sufficiently small we have $z(t) = 0$.

Hence there is a single pullback attracting positive trajectory $x*(\cdot)$.

for any two trajectories $x_1(\cdot)$ and $x_2(\cdot)$ by $x\_(t)$.

Now any trajectory $x(t, s, x_s)$ with $x_s > 0$ has$x(t, s, x_s) > \frac{3}{4} x_(t)$ for t large enough.

$x*(\cdot)$is attracting in (0,Є) as t→+∞.

The origin is unstable ‘downwards’ when $\lambda > 0$.

We have $0 \geq x(t) \geq u(t)$ where $u(t)$ is,

$$\dot{u} = \lambda[f(t) - \epsilon h(t)]u$$

as $t \to -\infty$ *the* $nu(t)$ *and we have* $x(t) \to 0$

The unstable trajectory when λ<0. The transformation $x \to -x, t \to -t$, for negative unstable trajectory;

$x*(\cdot)$ is attracting as t→+∞

4 Non-Autonomous Pitchfork Bifurcation

4.1 Definition

The system $\dot{x} = f(x, t, \lambda)$ undergoes a localised pitchfork bifurcation at $x = 0$, λ=0 if there exists $\lambda_0 > 0$ *and*$\epsilon > 0$ such that,

(i) for all $-\lambda_0 < \lambda \leq 0$ the zero solution is pullback attracting within $(-\epsilon, \epsilon)$

(ii) when $0 < \lambda < \lambda_0$ the zero solution is asymptotically unstable, and there exist bounded trajectories $x_\lambda^+(t)$ and $x_\lambda^-(t)$ that are pullback attracting in $(0, \epsilon)$ and $(-\epsilon, 0)$ respectively, $x_\lambda^\pm(t) \to 0$ as $\lambda \to 0$ uniformly on compact subsets of $\mathbb{R}$.

4.2 Example

Consider $x = 2y^2$, and satisfies the equation, $\dot{x} = 2\mu x - x^2$

With a bifurcation parameter $\lambda = 2\mu$, $\dot{x} = \lambda x - x^2$, where $x \geq 0$.

The original equation,

$$\dot{y} = H(y,t,\lambda)$$

i.e., Invariant under the transformation $y \to -y$ we set $x = y^2$

Consider, $\dot{x} = G(x,t,\lambda) = 2yH(y,t,\lambda)$

The existence of a non-autonomous pitchfork bifurcation under conditions of Theorem 3.1 in non-autonomous trans critical bifurcation.

5 Conclusion

Non-autonomous bifurcation theory extends classical insights by incorporating temporal variability, revealing rich dynamics such as branch switching and transient stability. Understanding these behaviours in trans critical and pitchfork cases remains key to advancing the theory of time-dependent dynamical systems.

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