



## Hausdorff Property of Transformation Graphs

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### Abstract

A Hausdorff graph  $G$  is a simple graph in which any two vertices  $u$  and  $v$  of  $G$  satisfy atleast one of the following conditions: (i) both  $u$  and  $v$  are isolated vertices (ii) either  $u$  or  $v$  is an isolated vertex (iii) there exists two non-adjacent edges  $e_1$  and  $e_2$  of  $G$  such that  $e_1$  is incident with  $u$  and  $e_2$  is incident with  $v$ . In this paper, we investigate Hausdorff property on transformation graphs.

**Key words:** Hausdorff Graph, Transformation Graph.

**AMS classification:** 05C76, 05C99.

## 1. Introduction

In [5], eight types of transformation graph were introduced and their basic properties were studied. Several authors have worked on these eight types of transformation graph separately. In [2], B. Wu, L. Zhang, Z. Zhang obtained a necessary and sufficient condition for  $G^{-++}$  to be hamiltonian.

Let  $G = (V(G), E(G))$  be a simple undirected graph and  $x, y, z$  be three variables taking values  $+$  or  $-$ . The transformation graph  $G^{xyz}$  is the graph having  $V(G) \cup E(G)$  as a vertex set, and two vertices  $\alpha$  and  $\beta$  of  $G^{xyz}$  are adjacent if and only if one of the following conditions holds: (i) for  $\alpha, \beta \in V(G)$ ,  $\alpha$  and  $\beta$  are adjacent in  $G$  if  $x = +$ ;  $\alpha$  and  $\beta$  are not adjacent in  $G$  if  $x = -$  (ii) for  $\alpha, \beta \in E(G)$ ,  $\alpha$  and  $\beta$  are adjacent in  $G$  if  $y = +$ ;  $\alpha$  and  $\beta$  are not adjacent in  $G$  if  $y = -$  (iii) for  $\alpha \in V(G)$ ,  $\beta \in E(G)$ ,  $\alpha$  and  $\beta$  are incident in  $G$  if  $z = +$ ;  $\alpha$  and  $\beta$  are not incident in  $G$  if  $z = -$ . Seena and Raji introduced Hausdorff properties in [3] and they discussed Hausdorff property of some derived graphs. A Hausdorff graph  $G$  is a simple graph in which any two vertices  $u$  and  $v$  of  $G$  satisfy atleast one of the following conditions: (i) both  $u$  and  $v$  are isolated vertices (ii) either  $u$  or  $v$  is an isolated vertex (iii) there exists two non-adjacent edges  $e_1$  and  $e_2$  of  $G$  such that  $e_1$

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is incident with  $u$  and  $e_2$  is incident with  $v$ . Terms not defined are used in the sense of [1]. We use the following theorems for proving main results.

**Theorem 1.1** [5] For a graph  $G$ ,  $G^{+yz} \cong G$  if and only if  $G$  is an empty graph.

**Theorem 1.2** [2] For a graph  $G$ ,  $G^{-++}$  is Hamiltonian if and only if  $|V(G)| \geq 3$ .

**Theorem 1.3** [3] Any Hamiltonian graph with more than 3 vertices is Hausdorff.

In this paper, we obtain results on  $G^{+++}$ ,  $G^{---}$ ,  $G^{-++}$  and  $G^{+--}$ .

## 2. Main Result

In this section, we obtain necessary and sufficient condition for  $G^{+++}$  to be Hausdorff, sufficient condition for  $G^{---}$  to be non-Hausdorff and sufficient condition for  $G^{-++}$  and  $G^{+--}$  to be Hausdorff.

**Theorem 2.1** The transformation graph  $G^{+++}$  of a graph  $G$  is Hausdorff if and only if  $G$  has no copy of  $K_2$  as a component.

Proof: Assume that  $G$  has no copy of  $K_2$  as a component. Let  $u_1, u_2$  be two distinct vertices of  $G^{+++}$ .

Case 1:  $u_1, u_2 \in V(G)$ .

(i)  $u_1$  and  $u_2$  are isolated in  $G$ . Then  $u_1$  and  $u_2$  are isolated in  $G^{+++}$ .

(ii)  $u_1$  or  $u_2$  is isolated in  $G$ . Then  $u_1$  or  $u_2$  is isolated in  $G^{+++}$ .

(iii)  $u_1$  and  $u_2$  are adjacent vertices of  $G$ . Let  $e_1 = u_1u_2$ . By hypothesis, there exists a vertex  $u_3$  such that  $u_3$  is adjacent to either  $u_1$  or  $u_2$ . Suppose that  $u_3$  is adjacent to  $u_1$  in  $G$ . Then clearly  $u_2e_1$  and  $u_1u_3$  are two non-adjacent edges of  $G^{+++}$ . Suppose that  $u_3$  is adjacent to  $u_2$  in  $G$ . Then clearly  $u_1e_1$  and  $u_2u_3$  are two non-adjacent edges of  $G^{+++}$ .

(iv)  $u_1$  and  $u_2$  are non adjacent vertices of  $G$ .

Since  $u_1$  and  $u_2$  are not isolated, there exists two distinct edges  $e_1$  and  $e_2$  such that  $e_1$  is incident with  $u_1$  and  $e_2$  is incident with  $u_2$ . Then  $u_1e_1$  and  $u_2e_2$  are two non-adjacent edges of  $G^{+++}$ .

Case 2:  $u_1, u_2 \in E(G)$ .

Since  $u_1 \neq u_2$ , there exists two distinct vertices  $u_3$  and  $u_4$  such that  $u_1$  is incident with  $u_3$  and  $u_2$  is incident with  $u_4$ . Then  $u_1u_3$  and  $u_2u_4$  are two non-adjacent edges of  $G^{+++}$ .

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Case 3:  $u_1 \in V(G)$  and  $u_2(\text{say } e_1) \in E(G)$ .

(i)  $e_1$  is incident with  $u_1$  in  $G$ .

Let  $e_1 = u_1u_3$  be an edge of  $G$ . By hypothesis, there exists an edge  $e_2$  different from  $e_1$  such that  $e_1$  and  $e_2$  are adjacent in  $G$ . Then  $e_1e_2$  and  $u_1u_3$  are two non-adjacent edges of  $G^{+++}$ .

(ii)  $e_1$  is not incident with  $u_1$  in  $G$ .

Suppose  $u_1$  is isolated in  $G$ . Then  $u_1$  is isolated in  $G^{+++}$ . Suppose  $u_1$  is not isolated in  $G$ . Then there exists an edge  $e_2$  incident with  $u_1$  in  $G$ . Let  $u_3$  be one endpoint of  $e_1$  in  $G$ . Then clearly  $u_1e_2$  and  $e_1u_3$  are two non-adjacent edges of  $G^{+++}$ . Thus  $G^{+++}$  is Hausdorff.

Conversely, Assume that  $G^{+++}$  is Hausdorff. Suppose  $K_2$  is one of the component of  $G$ . Let  $u_1, u_2$  be two distinct vertices of  $G^{+++}$ . Suppose  $u_1, u_2 \in V(K_2)$ . Let  $e_1 = u_1u_2$  be an edge of  $K_2$ . Then  $V(G^{+++}) \supseteq \{e_1, u_1, u_2\}$ . For these three vertices of  $G^{+++}$ , hausdorff property is not true. Therefore  $G$  has no copy of  $K_2$  as a component.

**Theorem 2.2** The transformation graph  $G^{---}$  of a graph  $G$  is not Hausdorff if  $G \cong K_{1,r} \cup K_1$  or  $G \cong K_{1,r} + e$  where  $e$  is an edge and  $r \geq 1$ .

Proof: Suppose  $G \cong K_{1,r} \cup K_1$ . Then  $G$  consists of an isolated vertex  $u_1$  and a vertex  $u_2$  such that  $u_2$  is adjacent to every other vertices of  $G$  other than  $u_1$ . Hence by the definition of the transformation graph  $G^{---}$ ,  $u_2$  is adjacent to  $u_1$  in  $G^{---}$  and no other vertices of  $G^{---}$  is adjacent to  $u_2$ . So  $\text{deg } u_2 = 1$  in  $G^{---}$ . Therefore  $G^{---}$  is not Hausdorff.

Suppose  $G \cong K_{1,r} + e$ . Then  $G$  consists of a vertex  $u$  such that  $u$  is adjacent to every other vertices of  $G$  and  $e$  is an edge of  $G$  not incident with  $u$ . Hence by the definition of the transformation graph  $G^{---}$ ,  $u$  is adjacent to  $e$  in  $G^{---}$  and no other vertices of  $G^{---}$  is adjacent to  $u$ . So  $\text{deg } u = 1$  in  $G^{---}$ . Therefore  $G^{---}$  is not Hausdorff.

**Remark 2.3** The converse of the above theorem need not be true. For a graph  $K_3^c$ ,  $G^{---}$  is not Hausdorff.

**Theorem 2.4** Let  $G$  be any graph of order  $n \geq 4$ . Then  $G^{-++}$  is Hausdorff.

Proof: By Theorem 1.2,  $G^{-++}$  is Hamiltonian. Hence by Theorem 1.3, it is Hausdorff.

**Theorem 2.5** If  $G$  is an empty graph then  $G^{+yz}$  is Hausdorff.

Proof: By Theorem 1.1,  $G^{+yz}$  is an empty graph. Hence it is Hausdorff.

**Theorem 2.6** Let  $G$  be any graph of order  $n \geq 4$ . If  $G$  has no copy of  $K_2$  as a component then  $G^{+--}$  is Hausdorff.

Proof: Let  $u_1$  and  $u_2$  be two distinct vertices of  $G^{+--}$ . Suppose  $G$  is an empty graph, then  $G^{+--}$  is an empty graph. Therefore  $G^{+--}$  is Hausdorff. Suppose  $G$  is not an empty graph.

Case 1:  $u_1, u_2 \in V(G)$ .

(i)  $u_1$  and  $u_2$  are isolated in  $G$ . Since  $G$  is not an empty graph, there exists at least one edge  $e_1$ . By hypothesis, there exists an edge  $e_2$  such that  $e_2$  is adjacent to  $e_1$ . Then  $u_1e_1$  and  $u_2e_2$  are two non-adjacent edges of  $G^{+--}$ .

(ii)  $u_1$  or  $u_2$  is isolated in  $G$ . Suppose  $u_1$  is isolated in  $G$ . Since  $u_2$  is not isolated in  $G$ , there exists a vertex  $u_3$  such that  $u_3$  is adjacent to  $u_2$ . Let  $e_1 = u_2u_3$ . Then  $u_1e_1$  and  $u_2u_3$  are two non-adjacent edges of  $G^{+--}$ .

(iii)  $u_1$  and  $u_2$  are adjacent vertices of  $G$ . Let  $e_1 = u_1u_2$ . By hypothesis, there exists another edge  $e_2$  of  $G$  which is incident with either  $u_1$  or  $u_2$ . Let us take  $e_2 = u_1u_3$ . Then  $u_1u_3$  and  $u_2e_2$  are two non-adjacent edges of  $G^{+--}$ .

(iv)  $u_1$  and  $u_2$  are non-adjacent vertices of  $G$ . Since  $u_1 \neq u_2$ , there exist two distinct edges  $e_1$  and  $e_2$  such that  $e_1$  is incident with  $u_1$  and  $e_2$  is incident with  $u_2$ . Then  $u_1e_2$  and  $u_2e_1$  are two non-adjacent edges of  $G^{+--}$ .

Case 2:  $u_1, u_2 \in E(G)$ .

Since  $u_1 \neq u_2$ , there exist two distinct vertices  $u_3$  and  $u_4$  of  $G$  such that  $u_1$  is incident with  $u_3$  and  $u_2$  is incident with  $u_4$ . Then  $u_1u_4$  and  $u_2u_3$  are two non-adjacent edges of  $G^{+--}$ .

Case 3:  $u_1 \in V(G), u_2 (= e_1) \in E(G)$ .

(i)  $e_1$  is incident with  $u_1$  in  $G$ . Let  $e_1 = u_1u_3$ . By hypothesis, there exists an edge  $e_2$  different from  $e_1$  such that  $e_2$  is incident with  $u_1$  or  $u_3$ . Let us suppose that  $e_2$  is adjacent to  $u_1$ . Let us take  $e_2 = u_1u_4$ . Then  $u_1u_3$  and  $e_1u_4$  are two non-adjacent edges of  $G^{+--}$ .

(ii)  $e_1$  is not incident with  $u_1$  in  $G$ . Suppose  $u_1$  is isolated in  $G$ . Let  $e_1 = u_3u_4$ . By hypothesis, there exists an edge  $e_2$  which is incident with either  $u_3$  or  $u_4$ . Let us take  $e_2 = u_4u_5$ . Then  $u_1e_2$  and  $e_1u_3$  are two non-adjacent edges of  $G^{+--}$ . Suppose  $u_1$  is not isolated in  $G$ . Then  $u_1$  is adjacent to a vertex  $u_3$  in  $G$ . Let us take  $e_2 = u_1u_3$ . Let  $u_4$  be an endpoint of  $e_1$  in  $G$ . Suppose  $e_1$  is adjacent to  $e_2$  in  $G$ . Then  $e_1$  is incident with  $u_3$ . Since  $n \geq 4$ , there exists a vertex  $u_5$  not incident with  $e_1$  other than  $u_1$ . Then clearly  $u_1u_3$  and  $e_1u_5$  are two non-adjacent edges of  $G^{+--}$ . Suppose  $e_1$  is not adjacent to  $e_2$  in  $G$ . Then  $u_1u_3$  and  $e_1e_2$  are two non-adjacent edges of  $G^{+--}$ .

Thus by all the above cases  $G^{+--}$  is Hausdorff.

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