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A Study of Hopf and Pitchfork Bifurcations in Planar Dynamical Systems

Geethalakshmi S[1], Kalaiyarasi S[2], Saraswathi D[3] and Vignesh K[4]

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Abstract

Hopf bifurcation in continuous and discrete planar systems is analyzed in this study along with the prerequisites for periodic solution emergence. It investigates the stability of both supercritical and subcritical conditions. Using examples and normal form analysis, the connection between Hopf and pitchfork bifurcations is also demonstrated.

Key Words: Bifurcation , Hopf bifurcation, Non-autonomous , Supercritical, planar system.

AMS Classification: 34D05, 34C23, 34C37, 34A34.

1 Introduction

1.1 Definitions

A Hopf (or) Poincare-Andronov-Hopf bifurcation is a local bifurcation in which a fixed point of a dynamical system loses stability as a pair of Complex Conjugate eigen values of linearization around the fixed point cross the imaginary axis of the complex plane.

The Hopf bifurcation theorem consider the planar system.

[1] Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur, Tamil Nadu, South India. Email:geethathiru126@gmail.com
[2] Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur, Tamil Nadu, South India. Email: skalaiarasi2@gmail.com
[3] Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur, Tamil Nadu, South India. Email: dswathisaranya@gmail.com
[4] Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur, Tamil Nadu, South India. Email: vigneshgokul210@gmail.com

$$\dot{X} = f_\mu(x, y)$$

$$\dot{y} = g_\mu(x, y)$$

Where μ is a parameter.

Suppose it has a fixed point $(x, y) = (x_0, y_0)$ which depend on μ.

Let the eigen values of the linearized system about this fixed point be given by,

$\lambda(\mu), \bar{\lambda}(\mu) = \alpha(\mu) \pm i\beta(\mu)$

For a certain value of μ, say $\mu = \mu_0$ the following conditions are satisfied:

1.2 Non Hyperbolicity Condition

$$\alpha(\mu_0) = 0, \ \beta(\mu_0) = w \neq o, where$$

$$sgn(\omega) = sgn\left[\left(\frac{\partial g\mu}{\partial x}\right)|_{\mu=\mu_0}(x_0, y_0)\right]$$

1.3 Transversality Condition

$$\frac{\partial \alpha(\mu)}{\partial \mu}|_{\mu=\mu_0} = d \neq 0$$

1.4 Genericity Condition

a≠o, where

$$a = \frac{1}{16}(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}) + \frac{1}{16\omega}(f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy})$$

$$with f_{xy} = \left(\frac{\partial^2 f_\mu}{\partial x \partial y}\right)|_{\mu=\mu_0}(x_0, y_0), \text{etc.}$$

Then a unique curve of periodic solutions bifurcates from the origin into the region $\mu > \mu_0$ if $ad < 0$ (or) $\mu < \mu_0$ if $ad > 0$.

The origin is a stable fixed point for $\mu > \mu_0$ and unstable fixed point for $\mu < \mu_0$ if $d < 0$ the periodic solutions are stable if the origin is unstable on the side of $\mu = \mu_0$ where the periodic solutions exists.

2 Hopf Bifurction for Maps

Consider the planar maps $F_\mu = (f_\mu, g_\mu): \mathbb{R}^2 \to \mathbb{R}^2$, with parameter μ and suppose it has a fixed point $(x, y) = (x_0, y_0)$, which may depend on μ.

Suppose that at this fixed point DF_μ has a complex conjugate pair of eigen values $\lambda(\mu)$,

$$\bar{\lambda}(\mu) = |\lambda(\mu)|e^{\pm i\omega(\mu)}$$

and that for a certain value of μ, say $\mu = \mu_0$, the following conditions are satisfied:

2.1 Non-Hyperbolicity Condition

$$|\lambda(\mu_0)| = 1$$

2.2 Non-Strong-Resonance Condition

$$\lambda^k(\mu_0) \neq 1 \text{ for } k = 1,2,3,4$$

2.3 Transversality Condition

$$\frac{d|\lambda(\mu)|}{d\mu}\Big|_{\mu=\mu_0} = d \neq 0$$

2.4 Genericity Condition

$$a \neq 0 \text{, where } a = -Re\left[\frac{(1-2e^{ic})e^{-2ic}}{1-e^{ic}} c_{11}c_{20}\right] - \frac{1}{2}|c_{11}|^2 - |c_{02}|^2 + Re\left(e^{-ic}C_{21}\right),$$

$$C = \omega(\mu_0),$$

$sgn\left(\omega(\mu_0)\right) = sgn[\left(\frac{\partial g_\mu}{\partial x}\right)|_{\mu=\mu_0}(x_0, y_0)]$ and

$$C_{20} = \frac{1}{8}\left[(f_{xx} - f_{yy} + 2g_{xy}) + i(g_{xx} - g_{yy} - 2f_{xy})\right]$$

$$C_{11} = \frac{1}{4}\left[(f_{xx} + f_{yy}) + i(g_{xx} + g_{yy})\right]$$

$$C_{02} = \frac{1}{8}\left[(f_{xx} - f_{yy} + 2g_{xy}) + i(g_{xx} - g_{yy} + 2f_{xy})\right]$$

$$C_{21} = \frac{1}{16}\left[(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}) + i(g_{xxx} + g_{xyy} - f_{xxy} - f_{yyy})\right]$$

Then an invariant simple closed curve bifurcates into either $\mu > \mu_0$ *or* $\mu < \mu_0$, depending on the signs of d *and* a.

This invariant circle is attracting if it bifurcates into the region of μ where the origin is unstable (a supercritical bifurcation) and repelling if it bifurcates into the region where the origin is stable (a subcritical bifurcation).

Hopf bifurcation for maps not structurally stable.

3 Hopf Bifurcation In Supercritical And Subcritical Conditions

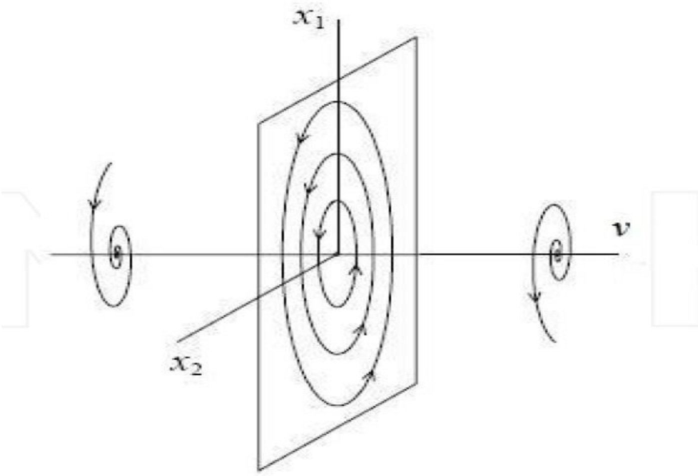
Consider the two dimensional system,

$$\frac{dx}{dt} = f(x, y, \tau)$$

$$\frac{dy}{dt} = g(x, y, \tau) \tag{3.1.1}$$

Where τ is the parameter and suppose that $(\bar{x}(\tau), \bar{y}(\tau))$ is the equilibrium point and $\alpha(\tau) \pm i\beta(\tau)$ are the eigen values of the Jacobian matrix which is evaluated at the equilibrium point.

Assume that the change in the stability of the equilibrium point occurs at $\tau = \tau^*$ where $\alpha(\tau^*) = 0$.



Hopf Bifurcation Diagram

First the system is transformed so that the equilibrium is at the origin and the parameter τ at $\tau^* = 0$ gives purely imaginary eigen values.

(3.1.1) is rewritten as,

$$\left.\begin{aligned} \frac{dx}{dt} &= a_{11}(\tau)x + a_{12}(\tau)y + f_1(x, y, \tau) \\ \frac{dy}{dt} &= a_{21}(\tau)x + a_{22}(\tau)y + g_1(x, y, \tau) \end{aligned}\right\} \tag{3.1.2}$$

The linearization of the system (3.1.1) about the origin is given by,

$$\frac{dx}{dt} = J(\tau)X$$

Where $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $J(\tau) = \begin{bmatrix} a_{11}(\tau) & a_{12}(\tau) \\ a_{21}(\tau) & a_{22}(\tau) \end{bmatrix}$

Is the Jacobian matrix evaluated at origin.

3.1 Definition Of Hopf Bifurcation

Let f_1, g_1 in system (3.1.2) have continuous third order partial derivatives in x and y. Suppose that the origin is an equilibrium point of (3.1.2) and that the Jacobian matrix J(τ)is valid for all sufficiently small |τ|. Assume that the eigen values of matrix J(τ) are $\alpha(\tau) \pm i\beta(\tau)$ where $\alpha(0) = 0$, $\beta(0) \neq 0$. Such the eigenvalues cross the imaginary axis.

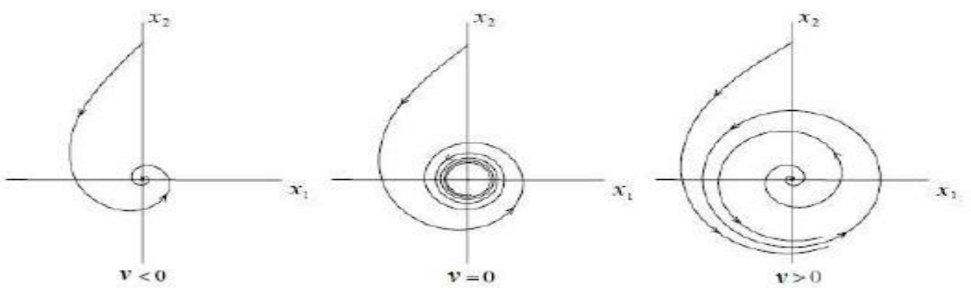
$$\frac{d\alpha}{d\tau}|_{\tau=0} \neq 0$$

Then in any open set μ containing the origin in $\mathbb{R}^2$ and for any $\tau_0 > 0$, there exists a value$\bar{\tau}$, $|\bar{\tau}| < \tau_0$. Such that the system of differential equations (3.1.2) has a periodic solution for $\tau = \bar{\tau}$in μ.

3.2 Definition

The bifurcation stated in the Hopf bifurcation theorem is called "Supercritical" if the equilibrium point (0,0) is asymptotically stable when $\tau = 0$ (at the bifurcation point) and it is called "subcritical" if the equilibrium point (0,0) is negatively asymptotically stable (as $t \to -\infty$) $when \tau = 0$.

In a supercritical Hopf bifurcation, the limit cycle grows out of the equilibrium point.



. Bifurcation diagram corresponding to Supercritical Hopf bifurcation

In right at the parameters of the Hopf bifurcation the limit cycle has zero amplitude, and this amplitude grows as the parameters move into the limit cycle.

In a subcritical Hopf bifurcation, there is an unstable limit cycle surrounding the equilibrium point and a stable limit cycle surrounding that. The unstable limit cycle shrinks down to the equilibrium point, which becomes unstable in the process.

For systems started near the equilibrium point, the result is a sudden change in behavior from to a stable focus, to large amplitude oscillations.

3.3 Example

Consider the two dimensional system,

$$\frac{\mathrm{dx}}{\mathrm{dt}} = x^2 - a$$

$$\frac{\mathrm{dy}}{\mathrm{dt}} = -(x^2 + 1)y \qquad (3.1.3)$$

Where a is a parameter.

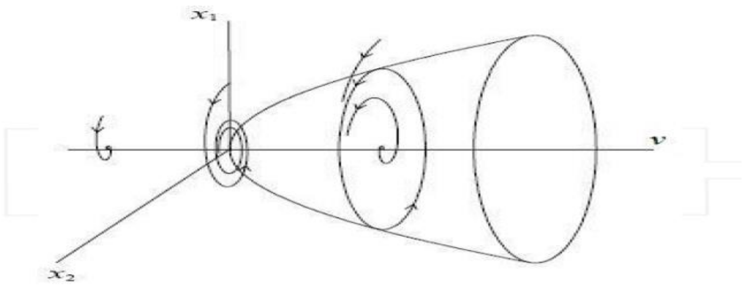
When the equilibrium points depending on parameter α;

If$\alpha < 0$, then there is no x, hence the system has no equilibrium points.

If$\alpha = 0$, then the system has exactly one equilibrium point at (0,0).

If$\alpha > 0$, then the system has two equilibrium points $(-\sqrt{\alpha}\,,0) and (\sqrt{\alpha}\,,0)$.

Supercritical Hopf Bifurcation



Then the Jacobian matrix is,

$$J = \begin{bmatrix} 2x & 0 \\ -2xy & -(x^2+1) \end{bmatrix}$$

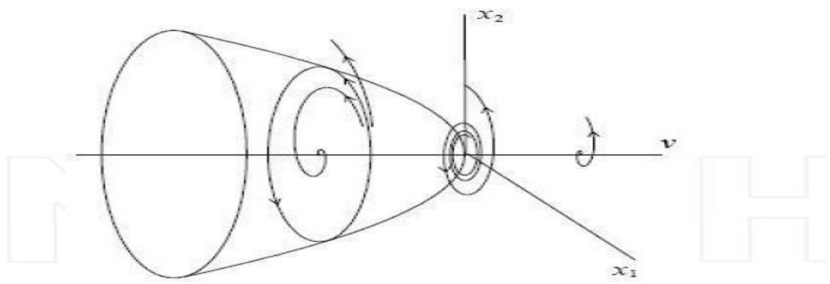
Where at the equilibrium points,

$$J = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

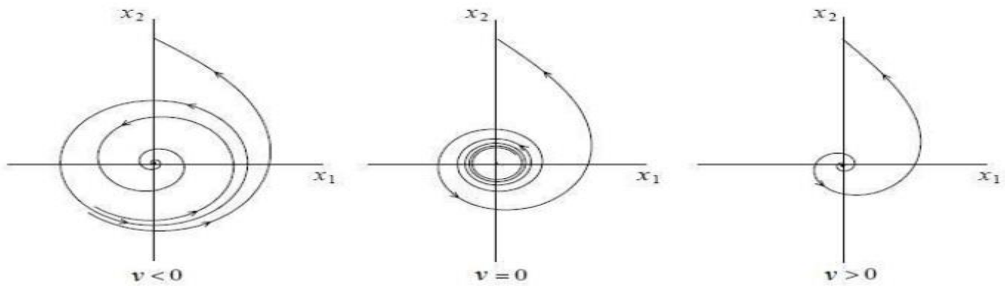
$J(-\sqrt{\alpha},0) = \begin{bmatrix} -2\sqrt{\alpha} & 0 \\ 0 & -\alpha-1 \end{bmatrix}$ and

$$J(\sqrt{\alpha},0) = \begin{bmatrix} -2\sqrt{\alpha} & 0 \\ 0 & -\alpha-1 \end{bmatrix}$$

For α= 0, there will be a line equilibrium and for $\alpha > 0$, the point $(-\sqrt{\alpha},0)$ is a sink and $(\sqrt{\alpha},0)$ is a saddle point so that $\alpha = 0$ is the bifurcation point for this differential equation system.



Subcritical Hopf Bifurcation



3.4 Example

Consider the two dimensional system,

$$\frac{dx}{dt} = y + \alpha x$$

$$\frac{dy}{dt} = -x + \alpha y \qquad (3.1.4)$$

where α is the bifurcation parameter the conditions of the Hopf bifurcation theorem hold.

In this system $f_1 and f_2$ are zero.

Then the Jacobian matrix is,

$$J = \begin{bmatrix} \alpha & 1 \\ -1 & \alpha \end{bmatrix}$$

For which the eigen values are $\lambda_{1,2} = \alpha \pm i$ where $Re\lambda(\alpha) = \alpha$and the imaginary part $Im\ \lambda(\alpha) = \pm 1$

It follows that $Re\lambda(0) = 0$ and $Im\ \lambda(0) \neq 0$ and also,

$$\left.\frac{dRe\lambda(\alpha)}{d\alpha}\right|_{\alpha=0} = 1 \neq 0$$

Hence, we conclude that there exists a periodic solutions for $\alpha = 0$ in every neighbourhood of origin.

4 Hopf Bifurcation With Pitchfork Bifurcation

The first order syst

$$\frac{du}{dt} = f(u, \mu) \quad (4.1.1)$$

must have dimension $n \geq 2$, only the planar case $n = 2$ when equation (4.1.1) becomes,

$$\left.\begin{aligned} \frac{du}{dt} &= f(u, v, \mu) \\ \frac{dv}{dt} &= g(u, v, \mu) \end{aligned}\right\} \quad (4.1.2)$$

where f and g are analytic functions of u ,v and μ.

The critical point $(\bar{u}(\mu), \bar{v}(\mu))$ of equation (4.1.2) corresponds to,

$$\left.\begin{aligned} f(\bar{u}(\mu), \bar{v}(\mu), \mu) &= 0 \\ g(\bar{u}(\mu), \bar{v}(\mu), \mu) &= 0 \end{aligned}\right\} \quad (4.1.3)$$

and its stability is determine by the eigenvalues $\lambda_1(\mu)$ and $\lambda_2(\mu)$of the Jacobian matrix,

$$A(\mu) = \begin{bmatrix} f_u(\bar{u}, \bar{v}, \mu) & f_v(\bar{u}, \bar{v}, \mu) \\ g_u(\bar{u}, \bar{v}, \mu) & g_v(\bar{u}, \bar{v}, \mu) \end{bmatrix} \quad (4.1.4)$$

One has for a Hopf bifurcation,

$$\lambda_1(\mu) = \overline{\lambda_2(\mu)} = \alpha(\mu) + i\beta(\mu) \quad (4.1.5)$$

Without loss of generality, the bifurcation point is $\mu = 0$, one has

$$\alpha(0) = 0,\ \beta(0) \neq 0 \quad (4.1.6)$$

Equation (4.1.6) implies that, in the neighbourhood of $\mu = 0$ $\det(\mu) \neq 0$, so that by the

Implies function theorem, $\bar{u}(\mu)$ and $\bar{v}(\mu)$ are analytic functions of μin a neighbourhood of $\mu = o$

$putting,$ $\quad \hat{u} \equiv u - \bar{u}$, $\hat{v} \equiv v - \bar{v}$ (4.1.7)

and expanding f and g in powers of $\hat{u}$ and $\hat{v}$ equation 4.1.2 becomes,

$$\begin{bmatrix} \frac{d\hat{u}}{dt} \\ \frac{d\hat{v}}{dt} \end{bmatrix} = A(\mu)\begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} + \begin{bmatrix} F(\bar{u}, \bar{v}, \mu) \\ G(\bar{u}, \bar{v}, \mu) \end{bmatrix} \quad (4.1.8)$$

Where F *and* G are $O(\hat{u}^2, \hat{v}^2, \hat{u}\hat{v})$ as $\hat{u}$ and $\hat{v} \to 0$ and are analytic functions of $\hat{u}$ and $\hat{v}$.

Let us suppose that $A(\mu)$ has the following canonical form

$$A(\mu) = \begin{bmatrix} \alpha(\mu) & \beta(\mu) \\ -\beta(\mu) & \alpha(\mu) \end{bmatrix}$$

So that equation (4.1.8) becomes,

$$\left.\begin{aligned} \frac{d\hat{u}}{dt} &= \alpha(\mu)\hat{u} + \beta(\mu)\hat{v} + F(\hat{u}, \hat{v}, \mu) \\ \frac{d\hat{v}}{dt} &= -\beta(\mu)\hat{u} + \alpha(\mu)\hat{v} + G(\hat{u}, \hat{v}, \mu) \end{aligned}\right\} \quad (4.1.9)$$

Equation (4.1.9) show that the origin $\hat{u} = 0$, $\hat{v} = 0$in the $\hat{u}, \hat{v}$- plane is a focus whose stability is determined by the sign of $\alpha(u)$.

Since $\alpha(0) = 0$, this stability changes, asμ passes through zero.

We consider,

$$Z = \hat{u} + i\hat{v} \quad (4.1.10)$$

So that equation (4.1.9) becomes,

$$\frac{dz}{dt} = [\alpha(\mu) - i\beta(\mu)]z + N(z, \bar{z}, \mu)$$

where$N(z, \bar{z}, \mu) \equiv F(\hat{u}, \hat{v}, \mu) + iG(\hat{u}, \hat{v}, \mu) \sim 0(|Z|^2)$ as $|Z| \to 0$

Using two successive near-identity analytic transformations of the form,

$$\xi = z + s(z, \bar{z}, \mu) \quad (4.1.11)$$

Where $s \sim O(|z|^2)$, as $|Z| \to 0$,(4.1.10) be reused to normal form,

$$\frac{d\xi}{dt} = [\alpha(\mu) - i\beta(\mu)]\xi + [\gamma(\mu) + i\delta(\mu)]|\xi|^2\xi + O(|\xi|^4) \quad (4.1.12)$$

where$\gamma(\mu)$and $\delta(\mu)$are analytic functions of μ.

Equation (4.1.12) implies that the nonlinear term $N(z, \bar{z}, \mu)$ in equation (4.1.10) can be transformed to remove all quadratic terms and all cubic terms except one, the term $|\xi|^2\xi$

$\therefore$The most dominant term producing resonance, as$|\xi| \to 0$.

4.1 Theorem

The first order equation,

$$\frac{d\xi}{dt} = [\alpha(\mu) - i\beta(\mu)]\xi + [\gamma(\mu) + i\delta(\mu)]|\xi|^2 + \xi \quad (4.1.13)$$

forμ in a neighbourhood of zero, has a family periodic solutions of period $2\pi/|\beta(0)|$, as$\mu \to 0$ which are stable (or unstable), when$\alpha(\mu) > 0$ (or) $\alpha(\mu) < 0$

Proof

Introduce the polar co-ordinates (R,ϕ) by

$$\xi = Re^{i\phi} \tag{4.1.14}$$

Equation (4.1.13) becomes,

$$\frac{dR}{dt} = \alpha(\mu)R + \gamma(\mu)R^3 \tag{4.1.15}$$

$$\frac{d\phi}{dt} = -\beta(\mu) + \delta(\mu)R^2 \tag{4.1.16}$$

Assuming that $\gamma(0) \neq 0$ so that $\gamma(\mu)$ is non-zero in a neighbourhood of $\mu = 0$.

Equation (4.1.15) has two critical points,

$$\bar{R} = 0 \tag{4.1.17}$$

$$\bar{R}^2 = -\frac{\alpha(\mu)}{\gamma(\mu)} \tag{4.1.18}$$

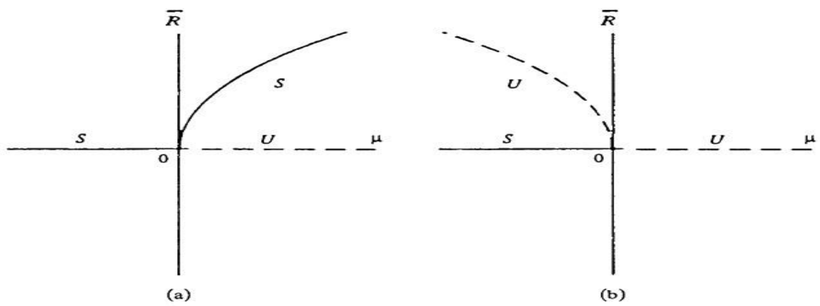
The stability of these solutions is determined by the sign of $\bar{\lambda}(\mu)$, where,

$$\bar{\lambda} = \frac{\partial}{\partial \bar{R}}(\alpha\bar{R} + \gamma\bar{R}^3) \text{ at } \alpha\bar{R} + \gamma\bar{R}^3 = 0 \tag{4.1.19}$$

(or) $$\bar{\lambda} = \begin{cases} \alpha, & \text{for } \bar{R} = 0 \\ -2\alpha, & \text{for } \bar{R}^2 = -\frac{\alpha}{\gamma} \end{cases} \tag{4.1.20}$$

Equation (4.1.20) shows two branches (4.1.17) & (4.1.18) has opposite stabilities.

(4.1.18) (nothing that $\alpha(\mu) \sim \alpha'(0)\mu$, as $\mu \to 0$ with condition $\alpha'(0) \pm 0$) is a pitchfork bifurcation.



BIFURCATION DIAGRAM A HOPF BIFURCATION

(a) Super critical and (b) subcritical

Equation (4.1.20) for branch (4.1.18) leads to,

$$\phi = \phi_0 + w(\mu)t \tag{4.1.21}$$

where $w(\mu) = -\beta - \frac{\delta\alpha}{\gamma}$

(4.1.18) corresponds to a periodic solution, with period $2\pi/|\,w(\mu)\,|$ (or) $2\pi/|\,\beta(0)\,|$, as $\mu \to 0$.

Therefore, in a supercritical Hopf bifurcation, as μvaries through the bifurcation point $\mu = 0, \bar{R} = 0$ changes from a stable focus to an unstable focus and leads to periodic solution. The Pitchfork bifurcation at $\mu = 0$, for equation (4.1.9) corresponds to Hopf bifurcation for the full system (4.1.7)

Thus Hopf bifurcation is generically a Pitchfork bifurcation.

5 Conclusion

This work offers a cohesive understanding of Hopf bifurcation in continuous and discrete systems. The nature of emerging periodic orbits is clarified by the differentiation between supercritical and subcritical actions. Furthermore, the theoretical foundation for examining local dynamical transitions is enhanced by the discovery that the Hopf bifurcation is a type of pitchfork bifurcation.

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