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Sharp bounds of hankel determinant on logarithmic coefficients for the family of bounded turning functions associated with exponential function

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Abstract

In this article, we examined a subclass of bounded turning functions, denoted as $\mathcal{R}_e$, which are associated with the exponential function. Sharp upper bounds for some of the initial coefficients are investigated for the class defined. Furthermore, we examined the logarithmic coefficient for this class. The main aim of this article is to obtain the sharp estimates of the second Hankel determinant with the logarithmic coefficient as entry for this class.

Key words: analytic; exponential function; hankel determinant; logarithmic coefficient.

AMS classification: Primary 30C45, 33C50; Secondary 30C80.

1 Introduction and Definition

Let $\mathcal{A}$ represent the collection of analytic functions defined within the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, which is normalized by the equation

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

Let $\mathcal{S} \subset \mathcal{A}$ denote the collection of normalized univalent functions. According to Köebe's $1/4-$theorem, for every univalent function f belonging to $\mathcal{S}$, there exists an inverse function f^{-1} that is defined at least within a disc of radius $1/4$, represented

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by the Taylor series of the form

$$f^{-1}(w) := w + \sum_{k=2}^{\infty} A_k w^k, \quad \left(|w| < \frac{1}{4}\right).$$

A function is considered bi-univalent in $\mathbb{D}$ if both f and its inverse f^{-1} are univalent in $\mathbb{D}$. In the recent past, several subclasses of analytic univalent functions and bi-univalent functions have been identified and researched; see [9, 10, 13, 26, 33, 34, 35, 36, 37, 38, 39] for further details.

The coefficient conjecture stating that $|a_k| \leq k$ for functions f belonging to $\mathcal{S}$, proposed by Bieberbach [7] in 1916, has drawn the attention of numerous researchers seeking to either validate or refute this assertion, until it was ultimately resolved by De Branges [8] in 1985. Throughout this timeframe, several significant subclasses of univalent functions were introduced and examined. The most recognized subfamilies include convex functions $\mathcal{K}$, starlike functions $\mathcal{S}^*$, and $\mathcal{R}$ of bounded turning functions which are defined as follows:

$$\mathcal{K} := \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D} \right\},$$

$$\mathcal{S}^* := \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D} \right\}$$

and

$$\mathcal{R} := \left\{ f \in \mathcal{A} : \Re\left(f'(z)\right) > 0, \quad z \in \mathbb{D} \right\}.$$

Let $\gamma \in (0, 1]$. A function $f \in \mathcal{A}$ is termed strongly starlike of order γ if it satisfies the condition:

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi\gamma}{2}, \quad z \in \mathbb{D}.$$

Furthermore, we define a function $f \in \mathcal{A}$ as strongly convex of order γ if

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi\gamma}{2}, \quad z \in \mathbb{D}.$$

The logarithmic coefficients γ_k of the function f belonging to the class $\mathcal{S}$ are significant in the field of estimation theory. They can be expressed using the formula:

$$\log\left(\frac{f(z)}{z}\right) = 2\sum_{k=1}^{\infty}\gamma_k z^k = F_f(z), \quad z \in \mathbb{D}.$$

De Branges [8] established that for $k \geq 1$,

$$\sum_{k=1}^{n} k(n-k+1)|\gamma_k|^2 \leq \sum_{k=1}^{n}\frac{n-k+1}{k},$$

holds true, and this equality is satisfied if and only if the function f is expressed as $\dfrac{z}{(1-e^{it}z)^2}$ for some θ in the real numbers. This inequality clearly presents the well-known Bieberbach - Robertson - Milin conjectures regarding the Taylor coefficients of functions f that belong to the class $\mathcal{S}$ in its most comprehensive form. In 2005, Kayumov [18] addressed Brennan's conjecture regarding conformal mappings by examining the logarithmic coefficients. For $k \geq 3$, the logarithmic coefficients problem appears to be more challenging. It is observed that the inequality $|\gamma_k| \leq \frac{1}{k}$ is valid for functions f belonging to $\mathcal{S}^*$, but it fails to hold for the entire class $\mathcal{S}$, even when considering an order of magnitude (refer to [13]). For notable investigations into logarithmic coefficients, see references [14, 12, 31].

If f is defined by equation (1), then its logarithmic coefficients are represented by

$$2\gamma_1 = a_2,$$

$$2\gamma_2 = a_3 - \frac{1}{2}a_2^2$$

and

$$2\gamma_3 = a_4 - a_2a_3 + \frac{1}{3}a_2^3.$$

For the specified functions f and g belonging to set $\mathcal{A}$, we denote the subordination of f to g (expressed as $f \prec g$) if there exists an analytic function ϑ defined in domain $\mathbb{D}$, subject to the condition that $\vartheta(0) = 0$ and $|\vartheta(z)| < 1$, such that the equation $f(z) = g(\vartheta(z))$ is satisfied. The function ϑ is referred to as a Schwarz function. Furthermore, it is established that if g in $\mathbb{D}$ is univalent, then

$$f(z) \prec g(z), \quad z \in \mathbb{D}$$

if and only if

$$f(0) = g(0) \quad and \quad f(\mathbb{D}) \subset g(\mathbb{D}).$$

In 1992, Ma and Minda [27] utilized the principle of subordination to propose a cohesive version of the class $\mathcal{S}^*(\phi)$ defined as follows:

$$\mathcal{S}^*(\phi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z), \quad z \in \mathbb{D} \right\}.$$

Assume ϕ is a univalent function satisfying $\phi'(0) > 0$ and $\phi > 0$. Additionally, the region $\phi(\mathbb{D})$ is star-shaped concerning the point $\phi(0) = 1$ and maintains symmetry along the real axis. In recent years, several sub-families of the set $\mathcal{S}$ have been investigated as specific examples of the class $\mathcal{S}^*(\phi)$. For example, if we select $\phi(z) = \frac{1+(1-2\delta)z}{1-z}$ with $0 \leq \delta < 1$, we obtain the class $\mathcal{S}^*(\delta) := \mathcal{S}^*\left(\frac{1+(1-2\delta)z}{1-z}\right)$ of starlike functions of order δ. It is remarked that $\mathcal{S}^* := \mathcal{S}^*\left(\frac{1+z}{1-z}\right)$ is essentially the traditional family of starlike functions. For further engaging related subclasses, consider examining [3, 5, 40].

The Hankel determinant $\mathcal{H}_{r,k}(f)$, where r and k are natural numbers, for a function $f \in \mathcal{S}$ of the series form (1), was introduced by Pommerenke in references [28, 29] as follows:

$$\mathcal{H}_{r,k}(f) = \begin{vmatrix} a_k & a_{k+1} & \cdots & a_{k+r-1} \\ a_{k+1} & a_{k+2} & \cdots & a_{k+r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+r-1} & a_{k+r} & \cdots & a_{k+2r-2} \end{vmatrix}$$

In existing literature, references to the Hankel determinant related to functions from the general family of univalent functions are scarce. In reference [15], it was demonstrated that $|\mathcal{H}_{2,k}(f)| \leq \lambda\sqrt{k}$, where $f \in \mathcal{S}$ and λ is a constant value. The quest to determine the precise boundaries of Hankel determinants within a specific class of functions has captivated many mathematicians. For instance, Janteng et al. [16, 17] computed the sharp bound of $|\mathcal{H}_{2,2}(f)|$ for the sub-families $\mathcal{K}$ and $\mathcal{S}^*$. It is evident from the equations presented in (10) that determining $|\mathcal{H}_{3,1}(f)|$ is significantly more complex than establishing the bound of $|\mathcal{H}_{2,2}(f)|$. In [4], Babalola investigated the bounds of the third-order Hankel determinant for the families of $\mathcal{K}$ and $\mathcal{S}^*$. Later, several authors [2, 6, 32, 42] obtained some interesting results on $|\mathcal{H}_{3,1}(f)|$ for certain sub-families of analytic and univalent functions. In recent years, some sharp bounds of the third-order Hankel determinant were obtained for several subclass of univalent functions. Kowalczyk et al. [23] and Lecko et al. [25] have verified that

$$|\mathcal{H}_{3,1}(f)| \leq \begin{cases} \dfrac{4}{135} & \text{if } f \in \mathcal{K} \\ \dfrac{1}{9} & \text{if } f \in \mathcal{S}^*\left(\dfrac{1}{2}\right). \end{cases}$$

In this context, $\mathcal{S}^*\left(\frac{1}{2}\right)$ refers to the family of starlike functions of order $\frac{1}{2}$. For more detailed contributions, see references [19, 30, 43, 44].

The notion of generalizing the Hankel determinant with logarithmic coefficients as its entries seems quite reasonable. Kowalczyk et al. [21, 22] were the first to introduce this concept in their works. With the logarithmic coefficient as a component, we determine that

$$\mathcal{H}_{r,k}(F_f/2) = \begin{vmatrix} \gamma_k & \gamma_{k+1} & \cdots & \gamma_{k+r-1} \\ \gamma_{k+1} & \gamma_{k+2} & \cdots & \gamma_{k+r} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{k+r-1} & \gamma_{k+r} & \cdots & \gamma_{k+2r-2} \end{vmatrix}$$

In particular, it is noted that

$$\mathcal{H}_{2,1}(F_f/2) = \begin{vmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_3 \end{vmatrix} = \gamma_1\gamma_3 - \gamma_2^2.$$

Let $f_t(z) := e^{-it}f(e^{it}z)$, $t \in \mathbb{R}$. It is observed that $\mathcal{H}_{2,1}(F_f/2)$ is invariant under rotation since we have

$$\mathcal{H}_{2,1}(F_{f_t}/2) = \frac{e^{4it}}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4\right) = e^{4it}\mathcal{H}_{2,1}(F_f/2).$$

A family of bounded turning functions associated with the modified sigmoid function was recently proposed by Khan et al. [20], expressed as

$$\mathcal{R}_{SG} := \left\{ f \in \mathcal{A} : f'(z) \prec \frac{2}{1+e^{-z}}, \quad z \in \mathbb{D} \right\}.$$

Inspired by the previously mentioned studies, this article presents the class $\mathcal{R}_{SG}$ defined as follows:

Definition 1.1 A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}_e$ if it satisfy the following subordination condition:

$$\mathcal{R}_e := \{ f \in \mathcal{A} : f'(z) \prec e^z, \quad z \in \mathbb{D} \}.$$

The exponential function $\phi(z) = e^z$ has a positive real part in $\mathbb{D}$ and an image domain $\phi(\mathbb{D}) := \{w \in \mathbb{C} : |\log w| < 1\}$.

In this article, we examined a subclass of bounded turning functions, denoted as $\mathcal{R}_e$, which are associated with the exponential function. Sharp upper bounds for some of the initial coefficients are investigated for the class defined. Furthermore, we examined the logarithmic coefficient for this class. The main aim of this article is to obtain the sharp estimates of the second Hankel determinant with the logarithmic coefficient as entry for this class. Recently, Sevtap Sümer Eker et al. [41] obtained the sharp bounds for the second Hankel determinant of logarithmic coefficients for strongly starlike and strongly convex functions.

2 Main Results

A function $h \in \mathcal{P}$ if and only if $\Re(h(z)) > 0$ for all z in the domain $\mathbb{D}$, represented by the series expansion

$$h(z) = 1 + \sum_{k=1}^{\infty} h_k z^k, \quad z \in \mathbb{D}. \tag{2}$$

Lemma 2.1 [11] If the function $h \in \mathcal{P}$ is given in the form (2), then

$$|h_k| \leq 2, \quad \forall \quad k \geq 1.$$

Lemma 2.2 [27] For any $\lambda \in \mathbb{C}$. If the function $h \in \mathcal{P}$ is given in the form (2), then

$$|h_2 - \lambda h_1^2| \leq \begin{cases} 2 - 4\lambda : & \lambda \leq 0, \\ 2 : & 0 \leq \lambda \leq 1, \\ 4\lambda - 2 : & \lambda \geq 1. \end{cases}$$

Lemma 2.3 [1] If the function $h \in \mathcal{P}$ is given in the form (2), with $0 \leq b \leq 1$ and $b(2b-1) \leq d \leq b$, then

$$|h_3 - 2bh_2h_1 + dh_1^3| \leq 2.$$

Lemma 2.4 [24] Let $h \in \mathcal{P}$. Then, for certain values of $x, \rho \in \bar{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$, it follows that

$$2h_2 = h_1^2 + (4 - h_1^2)x$$

and

$$4h_3 = h_1^3 + 2xh_1(4 - h_1^2) - x^2h_1(4 - h_1^2) + 2(4 - h_1^2)(1 - |x|^2)\rho.$$

Theorem 2.5 If the function $f \in \mathcal{A}$ of the form (1) belongs to the class $\mathcal{R}_e$ then

$$|a_2| \leq \frac{1}{2}, \tag{3}$$

$$|a_3| \leq \frac{1}{3} \tag{4}$$

and

$$|a_4| \leq \frac{1}{4}. \tag{5}$$

Proof: Assume the function $f \in \mathcal{A}$ takes the form (1) and is part of the class $\mathcal{R}_e$. As per Definition 1.1, there exists an analytic function $\omega(z)$ that satisfies the conditions of the Schwarz lemma (specifically, $\omega(0) = 0$ and $|\omega(z)| < 1$), such that

$$f'(z) = e^{\omega(z)}, \quad z \in \mathbb{D}. \tag{6}$$

Based on (1), we can conclude that

$$f'(z) = 1 + 2a_2z + 3a_3z^2 + 4a_4z^4 + \cdots . \tag{7}$$

Define a function h such that

$$h(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + h_1z + h_2z^2 + h_3z^3 + \cdots . \tag{8}$$

It follow that

$$\omega(z) = \frac{1}{2}h_1z + \left(\frac{1}{2}h_2 - \frac{1}{4}h_1^2\right)z^2 + \left(\frac{1}{2}h_3 - \frac{1}{2}h_1h_2 + \frac{1}{8}h_1^3\right)z^3 + \cdots .$$

Using the above series expansion, we obtain

$$e^{\omega(z)} = 1 + \frac{h_1}{2}z + \left(\frac{1}{2}h_2 - \frac{1}{8}h_1^2\right)z^2 + \left(\frac{1}{2}h_3 - \frac{1}{4}h_1h_2 + \frac{1}{48}h_1^3\right)z^3 + \cdots . \tag{9}$$

Now, comparing (7) and (9) leads to

$$a_2 = \frac{1}{4}h_1, \tag{10}$$

$$a_3 = \frac{1}{6}\left(h_2 - \frac{1}{4}h_1^2\right) \tag{11}$$

and

$$a_4 = \frac{1}{8}\left(h_3 - \frac{1}{2}h_1h_2 + \frac{1}{24}h_1^3\right). \tag{12}$$

By using Lemma 2.1 in (10) we get

$$|a_2| \le \frac{1}{2}.$$

Utilizing Lemma 2.2 and Lemma 2.3, in the context of equations (11) and (12) allows us to obtain equations (4) and (5).
The coefficient bounds of a_2, a_3 and a_4 are sharp. For that consider a function

$$f_k'(z) = e^{z^k}, \quad (k = 1, 2, 3). \tag{13}$$

This completes the proof of Theorem 2.5.

Theorem 2.6 If the function $f \in \mathcal{A}$ of the form (1) belongs to the class $\mathcal{R}_e$ then

$$|\gamma_1| \leq \frac{1}{4}, \tag{14}$$

$$|\gamma_2| \leq \frac{1}{6} \tag{15}$$

and

$$|\gamma_3| \leq \frac{1}{8}. \tag{16}$$

Proof: From (10), we get

$$\gamma_1 = \frac{h_1}{8}. \tag{17}$$

By using Lemma 2.1 in (17) we get

$$|\gamma_1| \leq \frac{1}{4}.$$

Now, from (10) and (11) we get

$$\gamma_2 = \frac{1}{12}\left(h_2 - \frac{21}{48}h_1^2\right). \tag{18}$$

By using Lemma 2.2 in (18) we get

$$|\gamma_2| \leq \frac{1}{6}.$$

Again, from (10), (11) and (12) we get

$$\gamma_3 = \frac{1}{16}\left(h_3 - \frac{5}{6}h_2h_1 + \frac{1}{6}h_1^3\right). \tag{19}$$

Now, from (19) we get

$$|\gamma_3| = \frac{1}{16}\left|h_3 - 2bh_1h_2 + dh_1^3\right|,$$

where

$$b = \frac{5}{12} \quad and \quad d = \frac{1}{6}.$$

We can see that

$$-\frac{5}{72} = b(2b-1) \leq d \leq b.$$

Therefore, by using Lemma 2.3, in (19) we get (16). This completes the proof of Theorem 2.6.

Theorem 2.7 If the function $f \in \mathcal{A}$ of the form (1) belongs to the class $\mathcal{R}_e$ then

$$\mathcal{H}_{2,1}(F_f/2) = |\gamma_1\gamma_2 - \gamma_2^2| \leq \frac{1}{36}. \tag{20}$$

Proof: Since $\mathcal{H}_{2,1}(F_f/2) = |\gamma_1\gamma_2 - \gamma_2^2|$ can be written as

$$\mathcal{H}_{2,1}(F_f/2) = |\gamma_1\gamma_2 - \gamma_2^2| = \frac{1}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4\right).$$

Now, from (10), (11) and (12), we get

$$\mathcal{H}_{2,1}(F_f/2) = \left|\frac{288h_3h_1 - 256h_2^2 - 16h_2h_1^2 - h_1^4}{36864}\right|.$$

Since $\mathcal{H}_{2,1}(F_f/2)$ is rotationally invariant, we may assume that $h_1 = h \in [0,2]$. Using Lemma 2.4 to express h_2 and h_3 in terms of $h_1 = h$, we obtain

$$\mathcal{H}_{2,1}(F_f/2) = \left|\frac{-h^4 + 8xh^2(4-h^2) - 72x^2h^2(4-h^2) - 64x^2(4-h^2)^2 + 144h(1-|x|^2)(4-h^2)\rho}{36864}\right|.$$

By replacing $|\rho| \leq 1$ and $|x| = y$, it follows that

$$\begin{aligned}\mathcal{H}_{2,1}(F_f/2) \leq& \frac{h^4 + 8yh^2(4-h^2) + 72y^2h^2(4-h^2) + 64y^2(4-h^2)^2 + 144h(1-y^2)(4-h^2)}{36864} \\ =& F(y,h).\end{aligned}$$

Differentiating with respect to y, we have

$$\frac{\partial F(y,h)}{\partial y} = \frac{(4-h^2)(h^2 + 18yh^2 + 64y - 16yh^2 - 36yh)}{4608}.$$

As $h \in [0,2]$, it is a simple exercise to show that $\dfrac{\partial F(y,h)}{\partial y}$ for $y \in [0,1]$. Thus, we have $F(h,y) \leq F(h,1)$. Putting $y=1$ gives

$$\mathcal{H}_{2,1}(F_f/2) = \frac{1024 - 192h^2 - 15h^4}{36864} := \Omega(h).$$

Differentiating with respect to h, we have

$$\Omega'(h) = -\frac{96h + 15h^3}{9216}.$$

Since $\Omega'(h) \leq 0$ for $h \in [0,2]$, we see that $\Omega(h)$ is a decreasing function, and it gives its maximum value at $h=0$. This yields

$$\mathcal{H}_{2,1}(F_f/2) = \frac{1024}{36864} = \frac{1}{36}.$$

Equality (20) is sharp for the function

$$f(z) = \int_0^z e^{t^2} dt = z + \frac{1}{3}z^2 + \frac{1}{5}z^5 + \cdots .$$

This completes the proof of Theorem 2.7.

3 Concluding remarks and observations

In this paper, the authors present a new family of bounded turning functions associated with the exponential function and obtained sharp upper bounds of some of the initial coefficients and logarithmic coefficient. The study also suggests that by employing q-calculus for values of $0 < q < 1$, along with functions that exhibit bounded boundary and bounded radius rotation, one can define a functional class. The results obtained in this study open new avenues for future research.

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