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Relation Between Class Number and Existence of Perfect Numbers

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Abstract

This work explores the relation between class numbers and perfect numbers through divisor sum formulas. It provides a structured derivation showing that even perfect numbers take the form $N = 2^m(2^{m+1} - 1)$ where $(2^{m+1} - 1)$is prime. The converse is also established, illustrating that such numbers are necessarily perfect. The study employs fundamental number-theoretic principles and factorization techniques to support the results.

Key words: Perfect numbers, Class numbers, Mersenne primes, Divisor sum formula, Even perfect numbers, Odd perfect numbers, Number theory, Prime distribution.

AMS classification:11A25 , 11A41 ,11R29.

1 Introduction

This paper investigates the connection between class numbers and the existence of perfect numbers. Using number-theoretic techniques, it derives formulas for the sum of divisors of integers and applies them to characterize even perfect numbers. A detailed proof of the classical result - every even perfect number is of the form $2^m(2^{m+1} - 1)$, where $(2^{m+1} - 1)$ is prime - is provided, along with its converse.

The usual criterion that a concept has been appropriately defined in an algebraic extension K of the rational number field Q is that the concept's most important characteristics and properties in Q have their counterpart in K. By this standard, the sum of divisors function has never been satisfactorily extended to an algebraic number field. The function is, of course, easily defined; however, the literature contains no attempts to define the concept in any algebraic number field other than $Q(\sqrt{-1})$, and this is almost certainly due to the difficulties experienced in proving the analogues in $Q(\sqrt{-1})$ of the best-known theorems involving the sum of

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divisors function. The sum of divisors function, in this respect, stands apart from the other familiar number-theoretic functions. The Euler phi-function and the number of divisors function, for example, are readily extended to a unique factorization domain having a finite number of units, for they are simply counting functions, and, the Moebius function was defined in $Q(\sqrt{-1})$ at the turn of the century. The analogue of each of the best-known results involving these functions, including, of course, the generalized Fermat theorem and the Moebius inversion formula have been shown to hold in those fields in which these functions have been defined.

The most widely-known theorem involving the sum of divisors function, apart, possibly, from the theorem establishing the multiplicative nature of the function, is the Euclid-Euler theorem characterizing the even perfect numbers in Q:

That the analogue of Theorem E-E has not been proven in an algebraic extension of Q (this is not quite true–see our comments below) is related to two problems which arise. TiLe first of these is that each of the concepts “positive”, “sum of divisors”, “Mersenne number”, “perfect number” and “even” must have a counterpart in the algebraic extension of Q which is reasonable and “natural” in some sense. A moment’s reflection reveals that there may be several reasonable ways to define each of these concepts, so that many combinations of the definitions are possible. This problem was discussed by Spira [3]whose definitions in $Q(\sqrt{-1})$ of “sum of divisors” and “Mersenne number” we have used in constructing our definitions in this paper. Spira proved an analogue of Euclid’s theorem stating the well-known sufficient condition that an even integer be a perfect number; using Spira’s definitions, the author of this paper subsequently proved [4] an analogue of Euler’s converse, subject to the restriction that the perfect numbers considered are primitive, i. e., not divisible by any other perfect number. (All perfect numbers in Q are, of course, primitive.) While these results come close to meeting the criteria that the concepts have been appropriately defined in an algebraic extension of Q, they appear to fail in one important respect. As W. D. Geyer [5] has properly mentioned in his review of Hausmann and Shapiro’s article [6], ”Spira has generalized the notion of a perfect number to elements of Z[i] in a certain artificial way...”. A perfect number had been defined as one whose divisor sum equals the product of the prime of least norm and the number itself. However, there is implicit in Spira’s definition a relationship which makes his definition of perfect number much less artificial than might appear. We will discuss this point more fully in section 5. The second problem encountered by the researchers who have examined the question of whether an analogue of Theorem E-E can be proved in $Q(\sqrt{-1})$ and which is, in fact, encountered in any unique factorization domain K having a finite number of units, is that Euler's proof ([7], p. 88, or see [8]) that all perfect numbers are of the form $2^{P-1}(2^p-1)$ for primes p and 2^p-1, and the variation of Euler's proof (apparently due to Dickson [9]) which appears in most introductory number theory texts, do not generalize to K. This is related to the fact that the sum of the divisors of a rational integer exceeds any partial sum of its divisors, whereas in K, the sum a of the divisors of an integer may be "closer" to zero than a partial sum of its divisors.

2 Premilinary

Theorem 2.1 Let $N = l_1^{m_1} l_2^{m_2} l_3^{m_3} \cdots l_n^{m_n}$,
here N is any positive integer, $l_1, l_2, \ldots, l_n$ are all distinct primes > 1 and $m_1, m_2, \ldots, m_n$ are integers ≥ 1.

Then, the sum of the divisors of N is

$$(1 + l_1 + l_1^2 + \cdots + l_1^{m_1})(1 + l_2 + l_2^2 + \cdots + l_2^{m_2}) \cdots (1 + l_n + \cdots + l_n^{m_n})$$

Proof Let $N = l_1^{m_1} X$, taking $X = l_2^{m_2}.l_2^{m_2}.l_2^{m_2}.l_3^{m_3} \cdots l_n^{m_n}$.
Let $1, X, P_1, P_2, \ldots, P_s$ be the distinct divisors of X.
We know $1, l_1, l_1^2, \ldots, l_1^{m_1}$ are the distinct divisors of $l_1^{m_1}$.
Let k be a divisor of N , then $\frac{N}{k} = R$ is also a divisor of N .

$$\frac{l_1^{m_1} X}{k} = R \qquad \text{[Here trivially } R \text{ is an integer]}$$

Let m_i be the maximum power of l_1 which can divide k,

$$[0 \leq m_i \leq m_1]$$

Then, $l_1^{m_i} \mid k$ and no more l_1 can divide $\frac{k}{l_1^{m_i}}$.

$$\Rightarrow \frac{l_1^{m_1} X}{k} = R \Rightarrow \frac{\frac{l_1^{m_1} X}{l_1^{m_i}}}{\frac{k}{l_1^{m_i}}} = R \Rightarrow \frac{l_1^{m_1 - m_i} X}{\left(\frac{k}{l_1^{m_i}}\right)} = R$$

$$\Rightarrow \left(\frac{k}{l_1^{m_i}}\right) \text{divides} l_1^{m_1-m_i}, X$$

From step (1), we know $(\frac{k}{l_1^{m_i}})$ and $(l_1^{m_1-m_i})$ are relatively prime.

$$\Rightarrow \frac{k}{l_1^{m_i}} \quad \text{can divide X only}$$

$$\frac{k}{l_1^{m_i}} = 1 \quad \text{or} \quad \frac{k}{l_1^{m_i}} = X \quad \text{or} \quad \frac{k}{l_1^{m_i}} = p_j \quad [1 \leq j \leq s]$$

Hence the possible divisors of N are

$$\begin{aligned}
&1,\ l_1,\ l_1^2,\ \ldots,\ l_1^{m_1}, && (1)\\
&X,\ l_1X,\ l_1^2X,\ \ldots,\ l_1^{m_1}X, && (2)\\
&p_1,\ p_1l_1,\ p_1l_1^2,\ \ldots,\ p_1l_1^{m_1}, && (3)\\
&p_2,\ p_2l_1,\ p_2l_1^2,\ \ldots,\ p_2l_1^{m_1}, && (4)\\
&\vdots && (5)\\
&p_s,\ l_1p_s,\ l_1^2p_s,\ \ldots,\ l_1^{m_1}p_s && (6)
\end{aligned}$$

Here above, it is trivial that all the above divisors are distinct since $1, X, p_1, \ldots, p_s, l_1, l_1^2, \ldots, l_1^{m_1}$ all are distinct.

Sum of the divisors of N (adding vertically)

$$\begin{aligned}
&= (1 + X + p_1 + \ldots + p_s) + (1 + X + p_1 + \ldots + p_s)l_1\\
&\qquad + (1 + X + p_1 + \ldots + p_s)l_1^2 + \cdots + (1 + X + p_1 + \ldots + p_s)l_1^{m_1}\\
&= (1 + X + p_1 + \ldots + p_s)(1 + l_1 + l_1^2 + \cdots + l_1^{m_1})
\end{aligned}$$

$$\text{Sum of divisors of } \; N = (1 + l_1 + \cdots + l_1^{m_1}) \times [\text{sum of divisors of } X]$$

$$= (1 + l_1 + \cdots + l_1^{m_1}) \times [\text{sum of divisors of}(l_2^{m_2}.l_3^{m_3} \cdots l_n^{m_n})] \qquad (7)$$

We know $X = l_2^{m_2} l_3^{m_3} \cdots l_n^{m_n} \quad \Rightarrow \quad X = l_2^{m_2}[Y]$, where $Y = l_3^{m_3} \cdots l_n^{m_n}$

Let $1, y, q_1, q_2 \ldots, q_t$ are divisors of Y.

Proceeding similarly as above,

Sum of divisors of $X = (1 + l_2 + l_2^2 + \cdots + l_2^{m_2})(1 + q_1 + q_2 + \cdots + q_t)$

$$= (1 + l_2 + l_2^2 + \cdots + l_2^{m_2})(\text{Sum of divisors of } \; l_3^{m_3}.l_4^{m_4} \cdots .l_n^{m_n}) \tag{8}$$

Now we know,
$Y = l_3^{m_3} l_4^{m_4} \cdots l_n^{m_n}$, $Y = l_3^{m_3}(Z)$
where $Z = l_3^{m_3} l_4^{m_4} \cdots l_n^{m_n}$.
Let $1, z, b_1, b_2, b_3, \ldots, b_k$ be possible divisors of Z.
Proceeding similarly,
Sum of divisors of
$Z = (1 + l_3 + l_3^2 + \cdots + l_3^{m_3})(1 + z + b_1 + b_2 + \cdots + b_k)$

$$= (1 + l_3 + l_3^2 + \cdots + l_3^{m_3})(\text{sum of divisors of } \; l_4^{m_4} l_5^{m_5} \cdots l_n^{m_n}) \tag{9}$$

Substituting (8) and (9) into (7), we have
Sum of divisors of $N = (1 + l_1 + l_1^2 + \cdots + l_1^{m_1})(1 + l_2 + l_2^2 + \cdots + l_2^{m_2})$

$$\times (1 + l_3 + l_3^2 + \cdots + l_3^{m_3}) \cdots (\text{sum of divisors of } l_4^{m_4} \cdots l_n^{m_n}) \tag{10}$$

Proceeding further, we will have
Sum of divisors of
$l_4^{m_4} \cdots l_n^{m_n} = (1 + l_4 + l_4^2 + \cdots + l_4^{m_4}) \cdot$ (sum of divisors of $l_5^{m_5} \cdots l_n^{m_n}$)
Sum of divisors of
$l_5^{m_5} \cdots l_n^{m_n} = (1 + l_5 + l_5^2 + \cdots + l_5^{m_5}) \cdot$ (sum of divisors of $l_6^{m_6} \cdots l_n^{m_n}$)

$$\vdots$$

Sum of divisors of $l_n^{m_n} = (1 + l_n + l_n^2 + \cdots + l_n^{m_n})$
$\therefore$ Substituting all into (10), we have

Sum of divisors of $N = (1 + l_1 + l_1^2 + \cdots + l_1^{m_1})(1 + l_2 + \cdots + l_2^{m_2}) \cdots (1 + l_n + \cdots + l_n^{m_n})$

3 Formula for Even Perfect Numbers

Theorem 3.1 If even number N is a perfect number, then N is of the form

$$N = 2^m(2^{m+1} - 1)$$

where $(2^m - 1)$ is prime.

Proof: Let N be an even perfect number.
Let m be the maximum power of 2 such that 2^m divides N.

$$N = 2^m X$$

where X is trivially an odd number and $m \geq 1$ (since N is even) .
Let $p_1, p_2, \ldots, p_s$ be the divisors of X other than 1 and X.
Sum of the divisors of N is given by

$$(1 + 2 + 2^2 + \cdots + 2^m)(1 + X + p_1 + \cdots + p_s)$$

$$2N = (1 + 2 + \cdots + 2^m)(1 + X + p_1 + \cdots + p_s)$$

$$2.2^m X = (1 + 2 + \cdots + 2^m)(1 + X + p_1 + \cdots + p_s)$$

$$(1 + 2 + 2^2 + \cdots + 2^m \text{is a odd number})$$

$$\therefore 1 + 2 + 2^2 + \cdots + 2^m = P_t, \quad \text{where} \quad X = P_r P_t.$$

similarly,

$$1 + X + p_1 + \cdots + p_s = \frac{2\dot{2}^m X}{1 + 2 + \cdots + 2^m}.$$

$$1 + X + p_1 + \cdots + p_s = 2.2^m P_r$$

now

$$\frac{1 + X + p_1 + \cdots + p_s}{p_r} = 2^{m+1}$$

$$P_t + \frac{1 + p_1 + \cdots + p_s}{p_r} = 2^{m+1}$$

$$\frac{1 + p_1 + \cdots + p_s}{p_r} = 2^{m+1} - P_t$$

$$= 2^{m+1} - [1 + 2 + 2^2 + \cdots + 2^m] = 2^{m+1} - \left[\frac{2^{m+1}}{2 - 1}\right] = 2^{m+1} - \left[2^{m+1} - 1\right]$$

$$\frac{1+p_1+\cdots+p_s}{p_r}=1$$

$$1+p_1+\cdots+p_s=p_r$$

We know that $\quad 1+2+\cdots+2^m=P_t$

$$\therefore P_t>1\Rightarrow\frac{X}{P_t}<X\Rightarrow P_r\neq X$$

$$\therefore P_r \text{ is one among } P_1,P_2,\ldots,P_s$$

(3) becomes

$$1+P_1+P_2+\cdots+P_t+P_r\cdots+P_s=P_s$$

$$\text{This is possible only if }\begin{cases}P_r=1\Rightarrow\text{Contradiction, since }P_t=X\\ \text{and }X\text{ divides }P_1,P_2,\ldots,P_s\text{ does not exist}\end{cases}$$

$\Rightarrow$ X has divisors 1 and X only $\Rightarrow$ X is prime

Using (4) $\quad 1+2+\cdots+2^m=X$

$\Rightarrow X=2^{m+1}-1$ which is prime

$\therefore \mathcal{N}=2^m\cdot X$

$\mathcal{N}=2^m(2^{m+1}-1)$ where $(2^{m+1}-1)$ is prime

Converse:

If an even number $\mathcal{N}$ is of the form

$$\mathcal{N}=2^m(2^{m+1}-1)$$

and $(2^{m+1}-1)$ is prime, then $\mathcal{N}$ should be perfect.

Proof:Let $\mathcal{N}=2^m(2^{m+1}-1)$

$$\mathcal{N}=2^m(X)\quad\text{where } X=2^{m+1}-1\text{ which is prime}$$

Sum of divisors of $\mathcal{N}$

$$=(1+2+\cdots+2^m)(1+X)=(2^{m+1}-1)(1+2^{m+1}-1)$$

$$=(2^{m+1}-1)(2^{m+1})=2^{m+1}(2^{m+1}-1)=2\cdot[2^m(2^{m+1}-1)]$$

sum of divisor of N =2N

$$\therefore N\ is\ perfect.$$

4 Condition for existence of odd perfect number N

Theorem 4.1 ***Condition for existence of odd perfect number N***

1. N can not be a perfect square number.

2. If $N = \ell_1^{m_1}\ell_2^{m_2}\ell_3^{m_3}\cdots\ell_n^{m_n}$, then only one among $m_1, m_2, \cdots, m_n$ can be odd all other are even.

3. Taking m_1 as odd, then $(\ell_1 + 1)$ is divisible by 2 only not by 4.

4. Taking m_1 as odd, then $(m_1 + 1)$ is divisible by 2 only not by 4.

5. Taking m_1 as odd, then ℓ_1 can not be the least among $\ell_1, \ell_2, \ell_3, \cdots, \ell_n$.

let N be an odd perfect number.

Let $N = \ell_1^{m_1}\ell_2^{m_2}\ell_3^{m_3}\cdots\ell_n^{m_n}$.

Here trivially $\ell_1, \ell_2, \ell_3, \cdots, \ell_n$ all are odd prime numbers only.

Sum of divisors of
$N = (1 + \ell_1 + \cdots + \ell_1^{m_1})(1 + \ell_2 + \cdots + \ell_2^{m_2})\cdots(1 + \ell_n + \cdots + \ell_n^{m_n})$

Therefore $2\ell_1^{m_1}\ell_2^{m_2}\ell_3^{m_3}\cdots\ell_n^{m_n}$

$$= (1 + \ell_1 + \cdots + \ell_1^{m_1})(1 + \ell_2 + \cdots + \ell_2^{m_2})\cdots(1 + \ell_n + \cdots + \ell_n^{m_n}) \tag{11}$$

Proof of Condition 1: N can not be a perfect square $m_1, m_2, \cdots, m_n$ all can not be even.

Let $m_1, m_2, \cdots, m_n$ all are even. Then $(1 + \ell_1 + \cdots + \ell_1^{m_1})(1 + \ell_2 + \cdots + \ell_2^{m_2})\cdots(1 + \ell_n + \cdots + \ell_n^{m_n})$ all will be odd.

$\therefore$ Their product also will be odd.

$\therefore$ R.H.S in (11) will be odd.

But L.H.S in (11) is even, Which Contradicts.

$\therefore$ N can not be a perfect square.

Proof of Condition 2: only one among $m_1, m_2, \cdots, m_n$ will be odd, all other should be even.

Let there exsits atleast two among $m_1, m_2, \cdots, m_n$ are odd. Let them be m_1 and m_2

$\therefore$ $(1 + \ell_1 + \cdots + \ell_1^{m_1})$ and $(1 + \ell_2 + \cdots + \ell_2^{m_2})$ both will be even.

[Since Sum of even number of odd numbers will be even]

$\therefore$ R.H.S in (11) will be divisible atleast by 2×2.

$\therefore$ R.H.S in (11) will be divisible atleast by 4.

But L.H.S in (11) is divisible only by 2, which Contradicts.

$\therefore$ only one among $m_1, m_2, \cdots, m_n$ can be odd, all others are even.

Equation (11) given,

$$2\ell_1^{m_1}\ell_2^{m_2}\ell_3^{m_3}\cdots\ell_n^{m_n} = (1 + \ell_1 + \cdots + \ell_1^{m_1})(1 + \ell_2 + \cdots + \ell_2^{m_2})\cdots(1 + \ell_n + \cdots + \ell_n^{m_n})$$

Here we have proved that only one among $m_1, m_2, \cdots, m_n$ can be odd all others are even.
Let m_1 is odd and all other $m_2, m_3 \cdots, m_n$ all are even.

$\therefore$ $m_1 + 1$ is even and $(1 + \ell_1 + \cdots + \ell_1^{m_1})$ will have even number of terms.

$$\therefore 1 + \ell_1 + \ell_1^2 + \cdots + \ell_1^{m_1} = 1 + \ell_1 + \ell_1^2(1 + \ell_1) + \cdots + \ell_1^{m_1 - 1}(1 + \ell_1)$$
$$= (1 + \ell_1)(1 + \ell_1^2 + \ell_1^4 + \cdots + \ell_1^{m_1 - 1)}$$

$$2\ell_1^{m_1}\ell_2^{m_2}\ell_3^{m_3}\cdots\ell_n^{m_n} = (1+\ell_1)(1 + \ell_1^2 + \ell_1^4 + \cdots + \ell_1^{m_1-1})(1+\ell_2+\cdots+\ell_2^{m_2})\cdots(1+\ell_n+\cdots+\ell_n^{m_n}) \tag{12}$$

Suppose $(\ell_1 + 1)$ is divisible by 4 then R.H.S in (12) is divisible atleast by 4.

But L.H.S is divisible by 2 which contradicts.

$\therefore$ If (odd perfect number) $N = \ell_1^{m_1}\ell_2^{m_2}\ell_3^{m_3}\cdots\ell_n^{m_n}$,[here m_1 is odd $m_2, m_3 \cdots m_n$ all are even]

then $(\ell_1 + 1)$ is divisible by 2 only not divisible by 4.

Equation (12) gives,

$$2\ell_1^{m_1}\ell_2^{m_2}\ell_3^{m_3}\cdots\ell_n^{m_n} = (1 + \ell_1)(1 + \ell_1^2 + \ell_1^4 + \cdots + \ell_1^{m_1-1})(1 + \ell_2 + \cdots + \ell_2^{m_2})\cdots$$
$$(1 + \ell_n + \cdots + \ell_n^{m_n})$$

$$2\ell_1({\ell_1}^2)^{(}\frac{m_1 - 1}{2})\ell_2^{m_2}\ell_3^{m_3}\cdots\ell_n^{m_n}$$

$$= (1+\ell_1)(1+\ell_1^2+\ell_1^4+\cdots+\ell_1^{m_1-1})(1+\ell_2+\cdots+\ell_2^{m_2})\cdots(1+\ell_n+\cdots+\ell_n^{m_n})$$

$$2\ell_1({\ell_1}^2)^{(}\frac{m_1-1}{2})\ell_2^{m_2}\ell_3^{m_3}\cdots\ell_n^{m_n} = (1+\ell_1)(1+L+L^2+\cdots+L^{(}\frac{m_1-1}{2}))$$
$$(1+\ell_2+\cdots+\ell_2^{m_2})\cdots(1+\ell_n+\cdots+\ell_n^{m_n})$$

Here suppose $(\frac{m_1-1}{2})$ is odd then $(1+L+L^2+\cdots+L^{(}\frac{m_1-1}{2}))$ is even. $\therefore$ $(1+\ell_1)(1+L+L^2+\cdots+L^{(}\frac{m_1-1}{2}))$ will be divisible atleast by 4.

But L.H.S is divisible by 2 only, which contradicts.

$\therefore (\frac{m_1-1}{2})$ is even.

$\therefore m_1-1$ is divisible atleast by 4.

$\therefore m_1+1$ is divisible by 2 only.

Here afterwards we can treat $(L)^{(}\frac{m_1-1}{2})$ as $(\ell_k^{m_k})$ and L as ℓ_k.

Here trivially m_k is even and ℓ_k is odd.

$\therefore N = \ell_1\ell_2^{m_2}\ell_3^{m_3}\cdots\ell_n^{m_n}$, where power of ℓ_1 in 1 and $m_2, m_3\cdots m_k\cdots m_n$ all are even.

$$2\ell_1\ell_2^{m_2}\ell_3^{m_3}\cdots\ell_n^{m_n} = (1+\ell_1)(1+\ell_2+\cdots+\ell_2^{m_2})\cdots$$
$$(1+\ell_k+\cdots+\ell_k^{m_k})\cdots(1+\ell_n+\cdots+\ell_n^{m_n})$$

$$2\ell_1\ell_2^{m_2}\ell_3^{m_3}\cdots\ell_n^{m_n} = (1+\ell_1)(1+\ell_2+\cdots+\ell_2^{m_2})\cdots(1+\ell_n+\cdots+\ell_n^{m_n})$$

We have

$$2\ell_1\ell_2^{m_2}\ell_3^{m_3}\cdots\ell_n^{m_n} = 2\ell_1X = (1+\ell_1)(1+\ell_2+\cdots+\ell_2^{m_2})\cdots(1+\ell_n+\cdots+\ell_n^{m_n})$$

$$\ell_1X = \frac{(1+\ell_1)}{2}(1+\ell_2+\cdots+\ell_2^{m_2})\cdots(1+\ell_n+\cdots+\ell_n^{m_n})$$

$$\frac{\ell_1X}{\frac{(1+\ell_1)}{2}} = (1+\ell_2+\cdots+\ell_2^{m_2})\cdots(1+\ell_n+\cdots+\ell_n^{m_n})$$

Here $\frac{(1+\ell_1)}{2}$ are relatively prime [since ℓ_1 is prime]

$\therefore \frac{(1+\ell_1)}{2}$ can divide X as

$\therefore \frac{(1+\ell_1)}{2}$ is a divisor of X where $X = \ell_2^{m_2}\ell_3^{m_3}\cdots\ell_n^{m_n}$.

$\frac{(1+\ell_1)}{2} \geq \ell_k$ (taking ℓ_k is the least among $\ell_2, \ell_3\cdots\ell_n$)

$(1+\ell_1) \geq 2\ell_k \Rightarrow \ell_1 \geq \ell_k+\ell_k-1$

$\therefore \ell_1 \geq \ell_k$

$\therefore \ell_1$ can not be the least among $\ell_1, \ell_2, \ell_3 \cdots \ell_n$.

We have $2N = 2\ell_1\ell_2^{m_2}\cdots\ell_n^{m_n} = (1+\ell_1)(1+\ell_2+\ell_2^{m_2})\cdots(1+\ell_n+\cdots+\ell_n^{m_n})$

we have $N = \ell_1\ell_2^{m_2}\cdots\ell_n^{m_n}$
Let $N = \ell_1 X$, here $X = \ell_2^{m_2}\ell_3^{m_3}\cdots\ell_n^{m_n}$

Let $1, X, p_1, p_2, \cdots, p_s$ be the divisors of X
Then we know Sum of divison of
$N = 2\ell_1\ell_2^{m_2}\cdots\ell_2^{m_2}\cdots\ell_n^{m_n} = (1+\ell_1)(1+X+p_1+\cdots+p_s)$

$2\ell_1 X = (1+\ell_1)(1+\ell_2+\cdots+\ell_2^{m_2})\cdots(1+\ell_n+\cdots+\ell_n^{m_n})$

$\ell_1 X = \dfrac{(1+\ell_1)}{2}(1+\ell_2+\cdots+\ell_2^{m_2})\cdots(1+\ell_n+\cdots+\ell_n^{m_n})$

$$\frac{\ell_1 X}{\left(\dfrac{(1+\ell_1)}{2}\right)} = (1+\ell_2+\cdots+\ell_2^{m_2})\cdots(1+\ell_n+\cdots+\ell_n^{m_n}) \tag{13}$$

Here $\left(\dfrac{(1+\ell_1)}{2}\right)$ and ℓ_1 are relatively prime $\quad [\therefore\ \ell_1$ is prime$]$

$\therefore\ \left(\dfrac{(1+\ell_1)}{2}\right)$ Can divide X and

Let $\dfrac{(1+\ell_1)}{2} = P_t$ and $\dfrac{X}{P_t} = P_r$
$1+\ell_1 = 2P_t$ and $X = P_r.P_t$

Substituting in (13)
$\dfrac{\ell_1 X}{P_t} = (1+\ell_2+\cdots+\ell_2^{m_2})\cdots(1+\ell_n+\cdots+\ell_n^{m_n}) = (1+X+p_1+\cdots+\rho_s)$
$P_r.\ell_1 = (1+\ell_2+\cdots+\ell_2^{m_2})\cdots(1+\ell_n+\cdots+\ell_n^{m_n}) = (1+X+p_1+\cdots+p_s)$

We have $1+\ell_1 = 2P_t$

$$P_r(1+\ell_1) = 2P_t.P_r$$

$$P_r + P_n\ell_1 = 2X$$

$$1+X+p_1+\cdots+p_s+p_r = 2X$$

$$1+p_1+\cdots+p_s+p_r = X \Rightarrow (1+X+p_1+\cdots+p_s) = p_r(p_t-1)$$

$\therefore$ we have

$$1+\ell_1 = 2p_t \tag{14}$$

$$1+X+p_1+\cdots+p_s = (1+\ell_2+\cdots+\ell_2^{m_2})\cdots(1+\ell_n+\cdots+\ell_n^{m_n}) = p_r\ell_1 \tag{15}$$

$$1+p_1+\cdots+p_s = p_r(p_t-1) \tag{16}$$

$$1+p_1+\cdots+p_s+p_r = X \tag{17}$$

We know that for any, $\ell_k^{m_k} \qquad (\text{here } \ell_k > 1)$

$$\ell_k^{m_k}-1 < \ell_k^{m_k} \Rightarrow \frac{\ell_k^{m_k}-1}{\ell_k-1} < \frac{\ell_k^{m_k}}{\ell_k-1}$$

$$1+\ell_1+\cdots+\ell_k^{m_k-1} < \ell_k^{m_k}\left(\frac{1}{\ell_k-1}\right)$$

$$\frac{1+\ell_1+\cdots+\ell_k^{m_k-1}}{\ell_k^{m_k}} < \frac{1}{\ell_k-1} \Rightarrow \frac{1+\ell_1+\cdots+\ell_k^{m_k-1}}{\ell_2^{m_2}} < \frac{1}{\ell_2-1}$$

similary we get

$$\frac{1+\ell_1+\cdots+\ell_k^{m_k-1}}{\ell_3^{m_3}} < \frac{1}{\ell_3-1}$$

$$\vdots$$

$$Also \;\; \frac{1}{\ell_1} < \frac{1}{\ell_1-1}$$

We know that

$$2\ell_1\ell_2^{m_2}\cdots\ell_n^{m_n} = (1+\ell_1)(1+\ell_2+\ell_2^{m_2})\cdots(1+\ell_n+\cdots+\ell_n^{m_n})$$

$$2 = \frac{(1+\ell_1)}{\ell_1}\frac{(1+\ell_2+\ell_2^{m_2})}{\ell_2^{m_2}}\frac{(1+\ell_3+\cdots+\ell_3^{m_3})}{\ell_3^{m_3}}\cdots\frac{(1+\ell_n+\cdots+\ell_n^{m_n})}{\ell_n^{m_n}}$$

$$2 < \left(1+\frac{1}{\ell_1}\right)\left(1+\frac{(1+\ell_2+\ell_2^{m_2})}{\ell_2^{m_2}}\right)\left(1+\frac{(1+\ell_3+\cdots+\ell_3^{m_3})}{\ell_3^{m_3}}\right)$$

$$\cdots\left(1+\frac{(1+\ell_n+\cdots+\ell_n^{m_n})}{\ell_n^{m_n}}\right)$$

$$2 < \left(1+\frac{1}{\ell_1-1}\right)\left(1+\frac{1}{\ell_2-1}\right)\cdots\left(1+\frac{1}{\ell_n-1}\right)$$

Let ℓ_2 is the least among $\ell_1,\ell_2,\ell_3,\cdots,\ell_n$

$\frac{1}{\ell_2 - 1}$ is the greatest among $\frac{1}{\ell_1 - 1}, \frac{1}{\ell_2 - 1}, \cdots, \frac{1}{\ell_n - 1}$

$$2 < \left(1 + \frac{1}{\ell_2 - 1}\right)\left(1 + \frac{1}{\ell_2 - 1}\right) \cdots \left(1 + \frac{1}{\ell_2 - 1}\right)$$

$$2 < \left[1 + \frac{1}{\ell_2 - 1}\right]^n \Rightarrow \left[1 + \frac{1}{\ell_2 - 1}\right]^n > 2$$

$$1 + \frac{1}{\ell_2 - 1} > 2^{\frac{1}{n}} \Rightarrow \frac{1}{\ell_2 - 1} > 2^{\frac{1}{n}} - 1 \Rightarrow \frac{1}{\ell_2 - 1} > Z - 1, \quad takingZ = 2^{\frac{1}{n}}$$

$$\frac{\ell_2 - 1}{1} > \frac{1}{Z - 1} \Rightarrow \ell_2 - 1 > \frac{1}{Z - 1} \Rightarrow \ell_2 > \frac{1}{Z - 1} + 1 \Rightarrow \ell_2 > \frac{1 + Z - 1}{Z - 1}$$

$$\ell_2 > \frac{Z}{Z - 1} \Rightarrow \ell_2 > \frac{2^{\frac{1}{n}}}{2^{\frac{1}{n}} - 1}$$

[here ℓ_2 is the least among $\ell_1, \ell_2, \ell_3, \cdots, \ell_n$]

Theorem 4.2 $ab < a + b$ if both a and b are less than 2 and both greater than 1.
Proof: Let $a = 1 + x_1$, Here $0 < x_1 < 1$
$b = 1 + x_2$ and $0 < x_2 < 1$

$$a + b = 2 + x_1 + x_2 \tag{18}$$

$ab = (1 + x_1)(1 + x_2)$

$$ab = 1 + (x_1 + x_2) + x_1 x_2 \tag{19}$$

$$(a + b) - (ab) = 1 - x_1 x_2 \tag{20}$$

Let $x_1 = \frac{p}{q}$ here $p < q \Rightarrow x_2 = \frac{r}{s}$ here $r < s$

$x_1 x_2 = \frac{pr}{qs}$, here $pr < qs$

$\therefore \quad x_1 x_2 < 1$

(20) becomes

$$(a + b) - (ab) > 0 \Rightarrow (a + b) > ab$$

$A_1 + A_2 + A_3 + \cdots + A_n > A_1 A_2 \cdots A_n$.
Let $(1 + x_1)(1 + x_2) \cdots (1 + x_n) = 2$

$[(1 + x_1)(1 + x_2) \cdots (1 + x_{n-1})]\,[1 + x_n] = 2$

P.Q $= 2$

Here trivally P is also such that $1 < P < 2$

[otherwise $P(1+x_n)$ will be greater than 2]

$$[(1+x_1)(1+x_2)\cdots(1+x_{n-1})]\,[1+x_n]$$

$$< [(1+x_1)\cdots(1+x_{n-1})] + [1+x_n]$$

Similary

$$[(1+x_1)(1+x_2)\cdots(1+x_{n-2})]\,[1+x_{n-1}]$$

$$< [(1+x_1)\cdots(1+x_{n-2})] + [1+x_{n-1}]$$

Proceeding further we will have

$$(1+x_1)(1+x_2)\cdots(1+x_{n-1}) < (1+x_1)+(1+x_2)+\cdots+(1+x_n)$$

$$A_1A_2\cdots A_n < A_1+A_2+\cdots+A_n \tag{21}$$

We have $(1+x_1)(1+x_2)\cdots(1+x_n) = 2$

$$1+x_1+x_2+\cdots+x_n+x_1x_2\cdots x_n = 2$$

$$x_1+x_2+\cdots+x_n < 1$$

Substituing in (21)

$$(1+x_1)(1+x_2)\cdots(1+x_n) < (1+x_1)(1+x_2)\cdots(1+x_n) < n+1 \tag{22}$$

Comparing (13) and (22) we have

$$x_1 = \frac{1}{\ell_1}; x_2 = \frac{(1+\ell_2+\cdots+\ell_2^{m_2-1})}{\ell_2^{m_2}}; \cdots ; x_n = \frac{(1+\ell_n+\cdots+\ell_n^{m_n-1})}{\ell_n^{m_n}}$$

Also $(1+x_1)+(1+x_2)+\cdots+(1+x_n) < n+1$
Let $(1+x_k)$ in the least among $(1+x_1),(1+x_2),\cdots,(1+x_n)$

$$n(1+x_k) < n+1 \Rightarrow n+n.x_k < n+1 \Rightarrow \frac{n(1+\ell_k+\cdots+\ell_k^{m_k-1})}{\ell_k^{m_k}} < 1$$

$$n(1+\ell_k+\cdots+\ell_k^{m_k-1}) < \ell_k^{m_k} \Rightarrow n < \ell_k \Rightarrow \ell_k > n$$

where $\left[\dfrac{1+\ell_k+\cdots+\ell_k^{m_k}}{\ell_k^{m_k}}\right]$ in the least among

$\left(\dfrac{1+\ell_1}{\ell_1}\right)\left(\dfrac{1+\ell_2+\ell_2^{m_2}}{\ell_2^{m_2}}\right)\cdots\left(\dfrac{1+\ell_n+\cdots+\ell_n^{m_n}}{\ell_n^{m_n}}\right)$

5 Conclusion

The characterization of even perfect numbers aids in cryptography, primality testing, and algebraic number theory. Understanding the link between class numbers and perfect numbers supports deeper investigations into prime distributions and contributes to the ongoing search for large Mersenne primes, which are central to identifying new perfect numbers.

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