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Bivariate Optimal Replacement Policies under Partial Product Process for Multistate Degenerative Systems

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Abstract

In this paper, we consider on a multistate degenerative system with k working states and l-failure states and study the maintenance problems under various bivariate replacement polices $(T, N), (T^+, N), (U, N), (U^-, N)$. The long-run average cost of a multistate degenerative system is calculated. Under the afore-mentioned bivariate replacement policies under partial product process optimality in inferred. In this study, the results developed are strengthened with numerical examples.

Key words: Partial Product Process, Replacement Policy, Renewal Reward Process and Virtual Repair Time.

AMS classification: 60K10, 90B25

1 Introduction

In practical scenario, because of the effect of aging and accumulated deterioration, the systems are categorized as degenerative in the sense that the succeeding work duration between failure become smaller and smaller, whereas the succeeding out-of-order maintenance repair durations are greater and greater. In otherwords, the successive operating times are dissipating, while the consecutive repair times are cycling to infinity with a stochastically increased average. To model a deteriorating system, with this kind of characteristics, Lam (1988) has introduced a Geometric processes and studied replacement problems. Stadje and Zuckerman (1990) have introduced a general monotone process repair model that generalised Lam's work other research works on the geometric process model include Stadje and Zuckerman (1992), Stanley (1993) for repair replacement models, Revathy (1997) for optimal

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replacement policies for stochastic system and Thangaraj and Rizwan (2001) for burn-in and maintenance models.

This study derives the long-run average costs for multistate degenerative systems under four bivariate replacement policies with partial product process:

- (T, N) policy : Replace the system after a fixed working age (T) or upon N-th failure
- (U, N) policy : Replace the system after cumulative repair time (U) or upon N-th failure
- (T^{+}, N) policy : Replace the system at the first failure point after cumulative operating time exceeds T or upon N-th failure
- (U^{-}, N) policy : Replace the system at the failure point just before total repair time exceeds U or upon N-th failure.

In reliability theory, one assumes that every component of the system works perfectly or is completely failed. This binary thinking does not hold up to scrutiny because a system can have more than just two states. For example, a microwave transmitter may be considered to be operating with full transmission range, operating with degraded transmission range, or completely failed. An example of failure contribution would be a special type of multistate system that has multiple distinct failures. Another example is a home security system that can be tampered with mechanically or electrically and also create false alarms when cats are detected in the house. Lesanovsky (1993) has provided a review of research on systems with dual failure modes. Zhang (1994), has introduced a bivariate optimal replacement policy for a two-state repairable system. Babu, Govindaraju and Rizwan (2018) introduced and studied replacement models where the consecutive repair time follow an increasing partial product process. Raajpandiyan, Syed Tahir Hussainy and Rizwan (2022) have studied optimal replacement models under partial product process.

In general, system may have one functional state and two distinct failure states. More generally, the system may have k-unique operating states and l-distinct failure states. For a multi-component system with $(k+l)$ states namely k-working states and l-failure states, this research focuses on a monotone process model. Such a model can be constructed in a number of ways to fit the description of a multistate degenerative system.

The rest of the paper is organized as follows. In Section 2, we give a general

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preliminaries. In Section 3, given model assumptions. We also present the monotone process model of a multi-component multistate system and the relevant results regrading their probability structure. In Section 4, We derive explicit expressions for the long-run average cost per unit time for this model under different bivariate replacement policies. Finally, a conclusion in given section 5.

2 Preliminaries

First, we provide a few definitions in this section. The model of a multi-component multistate system is then explained. In light of the system's current state, we additionally assess the conditional probabilities of the operating and failure times.

Definition 2.1 ***(Barlow and Proschan,1965)*** A random variable X is said to be stochastically smaller than anther random variable Y, if $P(X > \alpha) \leq P(Y > \alpha)$, for all real α. It is denoted by $X \leq_{st} Y$.

Definition 2.2 A stochastic process $X_n,\ n = 1, 2, \cdots$ is said to be stochastically increasing, if $X_n \leq_{st} X_{n+1}$, for $n = 1, 2, \cdots$.

Definition 2.3 ***(Shaked and Shanthikumar, 1994)*** A Markov process $X_n,\ n = 1, 2, \cdots$ with state space $0, 1, 2, \cdots$ is said to be Stochastically Monotone, if

$$(X_{n+1} \mid X_n = i_1) \leq_{st} (X_{n+1} \mid X_n = i),$$

for any $0 \leq i_1 \leq i_2$

Clearly, the stochastically monotone concept for a Markov process is defined for a Markov process and is based on the transition probabilities from one state to another state, conditioning on the former state. However, the stochastically monotone concept for a stochastic process defined here is for a general process and is based on the conditional distribution of the successive random variable in the process.

Definition 2.4 An integer valued random variable N is said to be a stopping time for the sequence of independent random variables $X_1, X_2, \cdots$, if the event $\{N = n\}$ is independent of $X_{n+1}, X_{n+2}, \cdots,$ for all $n = 1, 2, \cdots$.

Definition 2.5 ***(Babu, Govindaraju and Rizwan, 2018)*** Let $\{X_n, n = 1, 2, 3, \cdots\}$ be a sequence of independent and non-negative random variables and let $F(X)$ be the distribution function of X_1. Then $\{X_n, n = 1, 2, 3, \cdots\}$ is called partial product process, if the distribution function of X_{i+1} is $F(\alpha_i X)$ $(i = 1, 2, 3, \cdots)$, where $\alpha_i > 0$ are real constants and $\alpha_i = \alpha_0\alpha_1\alpha_2 \cdots \alpha_{i-1}$.

Definition 2.6 ***(Renewal process)*** If the sequence of nonnegative random variable $\{X_1, X_2, X_3, \cdots\}$ is independent and identically distributed, then the counting process $\{N(t), t \geq 0\}$ is said to be a renewal process.

Definition 2.7 A life distribution F is said to be new better then used in expectation, if

$$\int_0^\infty \overline{F}(t + x)dx \leq \overline{F}(t) \int_0^\infty \overline{F}(x)dx$$

for all $t \geq 0$.
To say that the life distribution of an item is new better then used in expectation is equivalent to saying that the mean life length of a new item is greater than the mean residual life length of a non-failed item of age $t > 0$.

Definition 2.8 At every failure point, a decision is taken whether it can be sent for repair. If the cumulative repair time after this repair is expected to exceed a threshold value δ, the repair need not be initiated at that failure time, such a fictitious repair time is called a ***Virtual Repair Time***.

Definition 2.9 A partial product process is called a ***decreasing partial product process***, if $\alpha_0 > 1$ and is called an ***increasing partial product process***, if $0 < \alpha_0 < 1$.

Remark 2.10 It is clear that if $\alpha_0 = 1$, then the Partial Product Process is a Renewal Process.

Remark 2.11 let $E(Y_1) = \mu$, $var(Y_1) = \sigma^2$. Then for $j = 1, 2, 3, \cdots$,
$E(Y_{j+1}) = \frac{\mu}{\beta_0^{2j-1}}$ and $var(Y_{j+1}) = \frac{\sigma^2}{\beta_0^{2j}}$, where $\beta_0 > 0$.

Theorem 2.12 ***(Wald's equation)*** If $X_1, X_2, X_3, \cdots$ are independent and identically distribution random variables having finite expectations and if N is the stopping time for $X_1, X_2, \cdots$ such that $E[N] < \infty$, then

$$E\left[\sum_{n=1}^{N} X_n\right] = E[N]E[X_1]$$

Theorem 2.13 ***(Wald's equation for partial product process)*** Suppose that $\{Y_n, n = 1, 2, 3, \cdots\}$ forms a partial product process with ratio β_0 and $E[Y_1] = \mu < \infty$, then for $t > 0$, we have

$$E\left[V_{\omega(t)+1}\right] = \mu E\left[1 + \sum_{j=2}^{\omega(t)+1} \frac{1}{\beta_0^{2j-2}}\right]$$

where $\omega(t)$ is the counting process which represents the number of occurrences of an event up to time t.

3 Model Assumptions

We shall now describe the system states. Consider a multistate system with $(k+l)$-states having k-working states and l-failure states. The system state at time t is given by

$$S(t) = \begin{cases} i & \text{if the system is in the i-th working state at time t } (i = 1, 2, \cdots, k) \\ k+j & \text{if the system is in the j-th working state at time t } (j = 1, 2, \cdots, l) \end{cases}$$

In a new system, the set of working states is $\Omega_1 = \{1, 2, \cdots, k\}$, and the set of failure states is $\Omega_2 = \{k+1, k+2, \cdots, k+l\}$ and the state space in $\Omega = \Omega_1 \cup \Omega_2$. In the beginning, suppose a brand new system at state 1 that is working is installed. It will be repaired, if the system fails. Let t_n be the completion time of the n-th repair, $n = 0, 1, \cdots$ with $t_0 = 0$ and let s_n be the time of occurrence of the n-th failure, $n = 1, 2, \cdots$. then

$$t_0 < s_1 < t_1 < ... < s_n < t_n < \cdots < s_{n+1} < \cdots,$$

we next describe the probability structure of the model.
Assume that the transition probability from working state i, $i = 1, 2, \cdots, k$, to failure

state $k+j, j=1,2,\cdots,l$, is given by

$$P(S(s_{n+1})=k+j \mid S(t_n)=i)=q_j$$

with $\sum_{j=1}^{l} q_j = 1$. Moreover, the transition probability from failure state $k+j, j = 1,2,\cdots,l$, to working state i, $i=1,2,\cdots,k$ is given by

$$P(S(t_n)=i \mid S(s_n)=k+j)=p_i$$

with $\sum_{j=1}^{k} P_j = 1$.

Let X_1 be the operating time of the system after installation. Let $X_n, n = 2,3,\cdots$ be the operating time of the system after (n-1)-st repair and $Y_n, n = 1,2,\cdots$ be the repair time after n-th failure. Assume that there exists a life distribution $U_i(t)$ and $a_i > 0, i = 1,2,\cdots,k$ such that

$$P(X_1 \leq t) = U_1(t) \tag{1}$$

and

$$P(X_2 \leq t \mid S(t_1)=i) = U_1(a_i t), \tag{2}$$

$i=1,2,\cdots,k$, where $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k$.

In gerenal, $i_j \, \epsilon \, \{1,2,\cdots,k\}$, we have

$$P(X_n \leq t \mid S(t_1)=i_1,\cdots,S(t_{n-1})=i_{n-1}) = U_1(a_i,\cdots,a_{i_{n-1}}t), \tag{3}$$

$j=1,2,...,n-1$.

Similarly, assume that there exist a life-time distribution $V_i(t)$ and $b_i > 0, i = 1,2,\cdots,l$ such that

$$P(Y_1 \leq t \mid S(s_1)=k+i) = V_1(b_1 t) \tag{4}$$

where $1 \geq b_1 \geq b_2 \geq \cdots \geq b_l > 0$ and in general, for $i_j \, \epsilon \, \{1,2,...,l\}$

$$P(Y_n \leq t \mid S(s_1)=k+i_1,\cdots,S(s_n)=k+i_n) = V_1(b_{i_1}\cdots b_{i_n}t) \tag{5}$$

In particular, if $a_1 = b_1 = 1, a_2 = \cdots = a_k = a'$ and $b_2 = \cdots = b_l = b'$ then the $(k+l)$-state system reduces to a two state system. In this case, the equations (3)

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and (5) become

$$\begin{aligned} P(X_n \leq t) &= U_1((a')^{n-1}t) \\ P(Y_n \leq t) &= V_1((b')^n t), \end{aligned}$$

respectively. Thus the sequence $X_n, n = 1, 2, \cdots$ from a partial product process with ratio $a' > 1$, while the sequence $Y_n, n = 1, 2, \cdots$ from with ratio $0 < b' < 1$. In this case, our model reduces to the model for the one component two state system introduced by Babu, Govindaraju and Rizwan (2018).

Remarks

For two working states $1 \leq i_1 < i_2 \leq k$, we have

$$(X_2 \mid S(t_1) = i_2) \leq_{st} (X_2 \mid S(t_1) = i_1).$$

so working state i_1 is better than working state i_2, in the sense that, the system in state i_1, has a stochastically large operating time than it does in state i_2. Consequently, the k-working states are arranged in decreasing order, such that state 1 is the best working state and state k is the worst working state. Similarly for two failure states $k + i_1,\ k + i_2$ such that $k + 1 \leq k + i_1 < k + i_2 \leq k + l$, we have

$$(Y_1 \mid S(s_1) = k + i_1) \leq_{st} (Y_1 \mid S(s_1) + k + i_2).$$

Because the system in state $k + i_1$ has a stochastically shorter repair time than it does in states $k + i_2$, the failure state $k + i_1$ is therefore superior to the failure state $k + i_2$. As a result, the l failure states are also sorted in decreasing order, with the best failure state being $k + 1$ and the worst failure state being $k + l$.
consider a monotone process model for a multistate system described in this section and make the following package of assumptions, A1 - A8.

- A1 At the beginning, a new simple repairable system is installed. The $(k + l)$-possible states exist for the system, where state $1, 2, \cdots, k$ indicate the first working state, the second working state,$\cdots$, k-th working state respectively, and state $(k + 1), (k + 2), \cdots, (k + l)$ indicate the first failure state, the second failure state,$\cdots$ and the l-th failure state of the system respectively. These failures are stochastically occurring and mutually exclusive.

A2 Whenever the system fails, it will be either repaired or replaced. The system will be replaced by an identical new one some times later.

A3 Let X_1 be the system's operating time after installation. Let X_n, $n = 2, 3, \cdots$ be the operating times of the system after the $(n-1)$-st repair in a cycle. The distribution of X_n is indicated by $F_n(.)$. Assume that $E(X_1) = \lambda > 0$.

Let X_{i+1} be the operating time after the i-th repair, for $i = 1, 2, 3, \cdots$. Then the distribution function of X_{i+1} is $F(\alpha_0^{2^{i-1}} x)$, where $\alpha_0(> 1)$ is a constant. Now

$$E(X_{i+1}) = \frac{\lambda}{\alpha_0^{2^{i-1}}}$$

for $i = 1, 2, 3, \cdots$. The successive operating time $X_n,\ n = 1, 2, 3, \cdots$ after repair constitute a decreasing partial product process.

A4 After the initial failure, let Y_1 represent the repair time and $G(y)$ represent the distribution function of Y_1. Assume that $E(Y_1) = \mu \geq 0$. When $\mu = 0$, it indicates that the anticipated repair time is negligibly small. After $(j+1)$-st failure, let Y_{j+1} be the repair time for $j = 1, 2, 3, \cdots$ and $G(\beta_0^{j-1} y)$ be the distribution function of Y_{j+1}, where $0 < \beta_0 \leq 1$ is a constant and $E(Y_{j+1}) = \frac{\mu}{\beta_0^{2^{j-1}}}$ for $j = 1, 2, 3, \cdots$.

The sequential repair durations from an increasing partial product process are $\{Y_j, j = 1, 2, 3, \cdots\}$.

A5 If the system in working state i is operating, then let the reward rate be r. The replacement cost comprises two parts one part is the basic replacement cost R and the other proportional to the replacement time z at rate c_p. If the system in failure state (k+i) is under repair, the repair cost is c. In otherwords, the replacement cost is given by $R + c_p Z$.

A6 Assume that $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k$ and $1 \geq b_1 \geq b_2 \geq \cdots \geq b_l > 0$.

A7 Assume that $F_n(t)$ be the cumulative distribution of $L_n = \sum_{i=1}^{n} X_i$ and $G_n(t)$ be the cumulative distribution of $M_n = \sum_{i=1}^{n} Y_i$.

A8 The working time X_n, the repair time Y_n and the replacement time Z, $(n = 1, 2, \cdots)$ are independent random variables.

4 Bivariate Replacement Policies

4.1 The Bivariate Policy (T, N)

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Here, we define and examine a bivariate replacement policy (T, N) under partial product process for the multistate degenerative system, where the system is replaced the at working age T or at the time of N-th failure, whichever occurs first. The problem is to determine an optimal replacement policy $(T, N)^*$ so that the long-run average cost per unit time is minimized.

The working age T of the system at time t is the cumulative life-time given by

$$T(t) = \begin{cases} t - M_n, & : \quad L_n + M_n \leqslant t < L_{n+1} + M_n \\ L_{n+1}, & : L_{n+1}, +M_n \leqslant t < L_{n+1} + M_{n+1}. \end{cases}$$

Initially let $L_n = \sum_{i=1}^{n} X_i$ and $M_n = \sum_{j=1}^{n} Y_j$ and $L_0 = M_0 = 0$.

Following Lam (2005), the distribution of the survival time X_n in $A3$ and the distribution of the repair time Y_n in $A4$ are given by

$$P(X_n \leq t) = \sum_{\sum_{i=1}^{k} j_i = n-1} \frac{(n-1)!}{i_1! i_2! \cdots i_k!} p_1^{i_1} \cdots p_k^{i_k} U(a_1^{i_1} \cdots a_k^{i_k}) t \tag{6}$$

where $j_1, j_2, \cdots, j_k \epsilon z^+$ and

$$P(Y_n \leq t) = \sum_{\sum_{i=1}^{l} j_i = n} \frac{(n)!}{j_1! j_2! \cdots j_l!} q_1^{j_1} \cdots q_l^{j_l} V(b_1^{j_1} \cdots b_l^{j_l}) t \tag{7}$$

where $j_1, j_2, \cdots, j_l \ \in \ Z^+$ and if $E(X_1) = \lambda$, then the mean survival time is

$$E(X_n) = \frac{\lambda}{\alpha_0^{2^{n-1}}} \tag{8}$$

for $n > 1$, where

$$a = \left(\sum_{i=1}^{k} \frac{p_i}{a_i} \right)^{-1} \tag{9}$$

and if $E(Y_1) = \mu$, then the mean repair time is

$$E(Y_n) = \frac{\mu}{\beta_0^{2^{n-1}}} \tag{10}$$

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for $n > 1$, where

$$b = \left(\sum_{j=1}^{l} \frac{q_j}{b_j} \right)^{-1}. \quad (11)$$

4.1.1 The length of a cycle and its mean

Then length of a cycle under the bivariate replacement policy (T, N) with partial product process is

$$w = \left(T + \sum_{i=1}^{\eta} Y_i \right) \chi(L_N > T) + \left(\sum_{i=1}^{N} X_i + \sum_{i=1}^{N-1} Y_i \right) \chi(L_N \leq T) + Z,$$

here the number of failures before the overall repair time above T is denotes by $\eta = 1, 2, \cdots, N-1$.

$$\chi_{(A)} = \begin{cases} 1 & : \text{if the even A occurs} \\ 0 & : \text{if the even A does not occur.} \end{cases}$$

Denotes the indicator function and $E[\chi(A)] = P(A)$.
From Leung (2006), we have

$$\begin{aligned} E[\chi(L_i \leq T < L_n] &= P(L_i \leq T < L_N) \\ &= P(L_i \leq T) - P(L_N \leq T) \\ &= F_i(T) - F_N(T). \end{aligned}$$

Lemma 4.1 The mean length of a cycle under the policy (T, N) is

$$E(W) = \int_0^T \overline{F}_N(u)du + \sum_{i=1}^{N-1} \frac{\mu}{\beta_0^{2^{i-1}}} F_i(T) + \tau \quad (12)$$

Proof: Examine

$$
\begin{aligned}
E(w) &= E\Big[\Big(T+\sum_{i=1}^{\eta} Y_i\Big)\chi(L_N > T)\Big] + E\Big[\Big(\sum_{i=1}^{N} X_i + \sum_{i=1}^{N-1} Y_i\Big)\chi(L_N \le T)\Big] + E(Z) \\
&= E[T\,\chi(L_N > T)] + E\Big[\Big(\sum_{i=1}^{\eta} Y_i\Big)\chi(L_N > T)\Big] \\
&\qquad + E\Big\{E\Big[\Big(\sum_{i=1}^{N} X_i + \sum_{i=1}^{N-1} Y_i\Big)\chi(L_N \le T) \ \Big|\ L_N = u\Big]\Big\} + E(Z) \\
&= T\overline{F}_N(T) + \sum_{i=1}^{N-1} \frac{\mu}{\beta_0^{2^{i-1}}} E\big[\chi(L_i \le T < L_N)\big] + \int_0^T u\,dF_N(u) \\
&\qquad + \int_0^T \sum_{i=1}^{N-1} Y_i\,dF_N(u) + \tau \\
&= T\overline{F}_N(T) + \sum_{i=1}^{N-1} \frac{\mu}{\beta_0^{2^{i-1}}} P(L_i < T < L_N) + \int_0^T u\,dF_N(u) + \sum_{i=1}^{N_1} \frac{\mu}{\beta_0^{2^{i-1}}} F_N(T) + \tau \\
&= T\overline{F}_N(T) + \int_0^T u\,dF_N(u) + \sum_{i=1}^{N_1} \frac{\mu}{\beta_0^{2^{i-1}}} [F_i(T) - F_N(T)] \\
&\quad + \sum_{i=1}^{N_1} \frac{\mu}{\beta_0^{2^{i-1}}} F_N(T) + \tau \\
&= \int_0^T \overline{F}_N(u)\,du + \sum_{i=1}^{N-1} \frac{\mu}{\beta_0^{2^{i-1}}} F_i(T) + \tau.
\end{aligned}
$$

as desired and this completes the proof of the lemma ■

4.1.2 The long-run average cost under policy (T, N)

The long-run average cost under policy (T, N) let take T_1 as the first replacement time. From this, we can derive $T_n (n \ge 2)$ as time taken between $(n-1)$-st and n-th replacement. The sequence $T_n,\ n = 1, 2, \cdots$, forms a renewal process. The inter-arrival time between two consecutive replacement is known as a renewal cycle. In the renewal reward theorem by Ross (1996), the long-run average cost per unit time under the multistate bivariate replacement policy (T, N) with partial product process is

$$\begin{aligned}\mathscr{C}(T,N) &= \frac{\text{the expected cost incurred in a cycle}}{\text{the expected length of a cycle}}\\ &= \frac{\left[\begin{array}{l} E\left\{\left(c\sum_{i=1}^{\eta}Y_i - rT\right)\chi(L_N > T)\right\} + c_pE(Z) \\ +E\left\{\left(c\sum_{i=1}^{N-1}Y_i - r\sum_{i=1}^{N}X_i\right)\chi(L_N \le T)\right\} + R \end{array}\right]}{E(W)}\end{aligned}$$

After simplifying, we assume to the following result using lemma 4.1.

Theorem 4.2 The long run average cost per unit time for a multistate degenerative system under the bivariate replacement policy (T,N) under partial product process for the model outlined in section 3 under the A1 through A8 is provided by.

$$\mathscr{C}(T,N) = \frac{\left(\sum_{i=1}^{N-1}\frac{\mu}{\beta_0^{2^{i-1}}}F_i(T)\right)c - \left(\int_0^T \overline{F}_N(u)du\right)r + R + c_p\tau}{\int_0^T \overline{F}_N(u)du + \sum_{i=1}^{N-1}\frac{\mu}{\beta_0^{2^{i-1}}}F_i(T) + \tau} \tag{13}$$

4.1.3 Deductions

The long-run average cost $\mathscr{C}(T,N)$ is a bivariate function in T and N. Obviously, when N is fixed, $\mathscr{C}(T,N)$ is a function of T. For fixed N=m, it can be written as

$$\mathscr{C}(T,N) = \mathscr{C}_m(T), m = 1,2,\cdots.$$

Thus, for a fixed m, we can find T_m^* by analytical or numerical methods such that $\mathscr{C}_m(T_m^*)$ is minimised. That is, when $N = 1,2,\cdots,m,\cdots$, we can find $T_1^*,T_2^*,\cdots,T_m^*,\cdots$, respectively, such that the corresponding, $C_1(T_1^*),C_2(T_2^*),\cdots,C_m(T_m^*),\cdots$, are minimised. Because the total lifetime of a multistate degenerative system is limited, the minimum of the long-run average cost per unit time exists. so, we can determine the minimum of the long-run average cost per unit time based on $C_1(T_1^*), C_2(T_2^*), \cdots, C_m(T_m^*), \cdots$.Then, if the is denoted by $\mathscr{C}_n(T_n^*)$, we obtain the bivariate optimal replacement policy $(T,N)^*$ such that

$$\begin{aligned} \mathscr{C}((T,N)^*) &= \min_n C_n(T_n^*) \\ &= \min_n[\min_T \mathscr{C}(T,N)] \\ &\leq \mathscr{C}(\infty, N) \\ &= C(N^*) \end{aligned}$$

The optimal policy $(T,N)^*$ is better than the optimal policy $N*$. moreover, under some mild conditions the optimal replacement policy N^* is better than the optimal policy T^*. So under the same conditions, an optimal policy $(T,N)^*$ is better than the optimal replacement policies N^* and T^*.

4.2 The Bivariate Policy (U,N)

Here, we define and examine a bivariate replacement policy (U,N) with partial product process for the multistate degenerative system, where the system is replaced when the system is at N-th failure or the overall time to repair is exceeds U, whichever comes first. The problem is to select an optimal replacement policy $(U,N)^*$ so that the long-run average cost per unit time is minimised.

4.2.1 The length of a cycle and its mean

The length of a cycle W under the bivariate replacement policy (U,N) with partial product process is

$$W = \left(\sum_{i=1}^{\eta} X_i + U\right)\chi(M_N > U) + \left(\sum_{i=1}^{N} X_i + \sum_{i=1}^{N-1} Y_i\right)\chi(M_N \leq U) + Z,$$

Where the number of failures before the overall repair time above U is denoted by $\eta = 1, 2, \cdots, N-\ 1$ and $\chi(.)$ denotes the indicator function.

$$\chi(A) = \begin{cases} 1 & : \text{if the event A occurs} \\ 0 & : \text{if the event A does not occur.} \end{cases}$$

Lemma 4.3 The mean length of the cycle under policy the (U,N) is

$$E(W) = \int_0^U \overline{G}_N(u)du + \frac{\lambda}{\alpha_0^{2^{N-1}}} G_N(U) + \sum_{i=1}^{N-1} \frac{\lambda}{\alpha_0^{2^{i-1}}} G_{i-1}(U) + \tau \tag{14}$$

Proof:

$$\begin{aligned} E(w) &= \left[\left(\sum_{i=1}^{\eta} X_i + U\right)\chi(M_N > U)\right] + E\left[\left(\sum_{i=1}^{N} X_i + \sum_{i=1}^{N-1} Y_i\right)\chi(M_N \le U)\right] \\ &\qquad + E(Z) \\ &= E\left[\sum_{i=1}^{\eta} X_{i\chi(M_N>U)}\right] + E[U_{\chi(M_N>U)}] \\ &\qquad + E\left\{E\left[\left(\sum_{i=1}^{N} X_i + \sum_{i=1}^{N-1} Y_i\right)\chi(M_N \le U) \mid M_N = u\right]\right\} + E(Z) \\ &= E\left(\sum_{i=1}^{N} X_i\right) E[\chi(M_N \le U) + \int_0^U u dG_N(u) \\ &\qquad + \sum_{i=1}^{N-1} E(X_i)E[\chi(M_{i-1} \le U < M_N)] + UE[\chi(M_N > U)] + \tau \\ &= \int_0^U u dG_N(u) + \sum_{i=1}^{N} \frac{\lambda}{\alpha_0^{2^{N-1}}} G_N(U) + \sum_{i=1}^{N-1} E(X_i)P(M_{i-1} \le U < M_N) \\ &\qquad + U\overline{G}_N(U) + \tau \\ &= \int_0^U u dG_N(u) + \sum_{i=1}^{N} \frac{\lambda}{\alpha_0^{2^{N-1}}} G_N(U) + \sum_{i=1}^{N-1} \frac{\lambda}{\alpha_0^{2^{i-1}}} [G_{i-1}(U) - G_N(U)] \\ &\qquad + U\overline{G}_N(U) + \tau, \end{aligned}$$

this provides equation (14) when simplified.

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4.2.2 The long-run average cost under policy (U, N)

The long-run average cost under policy (U, N) let take U_1 as the first replacement time. From this, we can derive $U_n (n \geq 2)$ as time taken between $(n-1)$-st and n-th replacement. The sequence $U_n,\ n = 1, 2, \cdots$ forms a renewal process. The inter-arrival time between two consecutive replacement is known as a renewal cycle. The long-run average cost per unit time under the multistate bivariate replacement policy (U, N) with partial product process is

$$\begin{aligned} \mathscr{C}(U,N) &= \frac{\text{the expected cost incurred in a cycle}}{\text{the expected length of a cycle}} \\ &= \frac{\left[\begin{array}{l} E\left\{ \left(cU - r\sum_{i=1}^{\eta} X_i \right) \chi(M_N > U) \right\} + c_p E(Z) \\ +E\left\{ \left(c\sum_{i=1}^{N-1} Y_i - r\sum_{i=1}^{N} X_i \right) \chi(M_N \leq U) \right\} + R \end{array} \right]}{E(W)} \end{aligned}$$

After simplifying, we assume to the following result using lemma 4.2.

Theorem 4.4 The long run average cost per unit time for a multistate degenerative system under the bivariate replacement policy (U,N) with partial product process for the model outlined in section 3 under the A1 through A8 is provided by.

$$\mathscr{C}(T,N) = \frac{\left(\int_0^U \overline{G}_N(u)du \right) c - \left(\frac{\lambda}{\alpha_0^{2^{N-1}}} G_N(U) + \sum_{i=1}^{N-1} \frac{\lambda}{\alpha_0^{2^{i-1}}} G_{i-1} \right) r + R + c_p\tau}{\frac{\lambda}{\alpha_0^{2^{N-1}}} G_N(U) + \sum_{i=1}^{N-1} \frac{\lambda}{\alpha_0^{2^{i-1}}} G_{i-1} + \int_0^U \overline{G}_N(u)du + \tau} \tag{15}$$

4.2.3 Deductions

The long-run average cost $\mathscr{C}(U, N)$ is a bivariate function in U and N. Obviously, when N is fixed, $\mathscr{C}(U, N)$ is a function of U. For fixed N=m, it can be written as

$$\mathscr{C}(U,N) = \mathscr{C}_m(U), m = 1, 2, \cdots$$

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Thus, for a fixed m, we can find U_m^* by analytical or numerical methods such that $\mathscr{C}_m(U_m^*)$ is minimised. That is, when $N = 1, 2, \cdots, m, \cdots$, we can find $U_1^*, U_2^*, \cdots, U_m^*, \cdots$, respectively, such that the corresponding, $C_1(U_1^*), C_2(U_2^*), \cdots, C_m(U_m^*), \cdots$, are minimised. Because the total lifetime of a multistate degenerative system is limited, the minimum of the long-run average cost per unit time exists. so, we can determine the minimum of the long-run average cost per unit time based on $C_1(U_1^*), C_2(U_2^*), \cdots, C_m(U_m^*), \cdots$.Then, if the is denoted by $\mathscr{C}_n(U_n^*)$, we obtain the bivariate optimal replacement policy $(U, N)^*$ such that

$$\begin{aligned}
\mathscr{C}((U,N)^*) &= \min_m C_m(U_m^*) \\
&= [\min_U \mathscr{C}(U,N)] \\
&\leq \mathscr{C}(\infty, N) \\
&= C(N^*)
\end{aligned}$$

The optimal policy $(U, N)^*$ is better than the optimal policy $N*$. moreover, under some mild conditions the optimal replacement policy N^* is better than the optimal policy U^*. So under the same conditions, an optimal policy $(U, N)^*$ is better than the optimal replacement policies N^* and U^*.

4.3 The Bivariate Policy (T^+, N)

It is a policy where the multistate degenerative system, where system replaced at the initial failure point upon cumulative operating time exceeding T or at occurrence of the N-th failure, which ever comes first, The method of replacement at the first failure point one the total operating time is greater than a given value is employed in Muth (1977).

4.3.1 The length of a cycle and its mean

The length of a cycle W under the bivariate replacement policy (T^+, N) with partial product process is

$$W = \left(\sum_{i=1}^{\eta} X_i + \sum_{i=1}^{\eta} Y_{i-1}\right)\chi(L_N > T) + \left(\sum_{i=1}^{N} X_i + \sum_{i=1}^{N-1} Y_i\right)\chi(L_N \leq T) + Z,$$

Where the number of failures before the overall repair time above T is denoted by $\eta = 1, 2, \cdots, N-1$.

$$\begin{aligned} P(\eta = j) &= P(X_1 \leq T, X_2 \leq T, \cdots, X_{\eta-1} \leq T, X_\eta > T); j = 1, 2, \cdots \\ &= \overline{F}(T) F^{j-1}(T). \end{aligned}$$

since η is a random variable,

$$\begin{aligned} E(\eta - 1) &= \sum_{j=1}^{\infty} (j-1) P(\eta = j) \\ &= \overline{F}(T) \sum_{j=1}^{\infty} (j-1) F^{j-1}(T) \\ &= \frac{F(T)}{\overline{F}(T)}. \end{aligned}$$

Lemma 4.5 The mean length of the cycle under policy the (T^+, N) is

$$\begin{aligned} E(W) &= \sum_{i=1}^{N-1} \frac{\mu}{\beta_0^{2^{i-1}}} [(1-b) F_N(T) + b F_i(T)] + \frac{F(T)}{\overline{F}(T)} \sum_{i=1}^{N-1} \frac{\lambda}{\alpha_0^{2^{i-1}}} [(F_i(T) - F_N(T)] \\ &\quad + \int_0^T u dF_N(u) + \tau \end{aligned} \tag{16}$$

Proof: Examine

$$\begin{aligned} E(w) &= E\left[\left(\sum_{i=1}^{N} X_i + \sum_{i=1}^{N-1} Y_i\right) \chi(L_N \leq T)\right] + E\left[\left(\sum_{i=1}^{\eta} X_i + \sum_{i=1}^{\eta} Y_{i-1}\right) \chi(L_N > T)\right] \\ &\quad + E(Z) \\ &= E\left\{E\left[\left(\sum_{i=1}^{N} X_i + \sum_{i=1}^{N-1} Y_i\right) \chi(L_N \leq T) \mid L_N = u\right]\right\} \\ &\quad + E\left[\sum_{i=1}^{\eta} X_i \chi(L_N > T)\right] + E\left[\left(\sum_{i=1}^{\eta} Y_{i-1}\right) \chi(L_N > T)\right] + E(Z) \end{aligned}$$

$$\begin{aligned}
&= \int_o^T u dF_N(u) + \int_o^T \sum_{i=1}^{N-1} E(Y_i) dF_N(u) + \sum_{i=1}^{N-1} \frac{\mu}{\beta_0^{2^{i-1}}} P(L_i \leq T < L_N) \\
&\quad + \sum_{i=1}^{N-1} E(X_i \mid \eta = N-1) P(L_i \leq T < L_N) \\
&\quad + \sum_{i=1}^{N-1} E(Y_{i-1}) E[\chi(L_i \leq T < L_N)] + \tau \\
&= \int_o^T u dF_N(u) + \sum_{i=1}^{N-1} \frac{\mu}{\beta_0^{2^{i-1}}} F_N(T) + \frac{F(T)}{\overline{F}(T)} \sum_{i=1}^{N-1} \frac{\lambda}{\alpha_0^{2^{i-1}}} [F_i(T) - F_N(T)] \\
&\quad + \sum_{i=1}^{N-1} \frac{\mu}{\beta_0^{2^{i-1}}} [F_i(T) - F_N(T)] + \tau,
\end{aligned}$$

This provides equation (16) when simplified.

4.3.2 The long-run average cost under policy (T^+, N)

The long-run average cost per unit time under the multistate bivariate replacement policy (T^+, N) under partial product process is

$$\begin{aligned}
\mathscr{C}(T^+, N) &= \frac{\text{the expected cost incurred in a cycle}}{\text{the expected length of a cycle}} \\
&= \frac{\left[\begin{array}{c} E\left\{ \left(c \sum_{i=1}^{\eta} Y_i - r \sum_{i=1}^{\eta} X_i \right) \chi(L_N > T) \right\} + c_p E(Z) \\ +E\left\{ \left(c \sum_{i=1}^{N-1} Y_i - r \sum_{i=1}^{N} X_i \right) \chi(L_N \leq T) \right\} + R \end{array} \right]}{E(W)}
\end{aligned}$$

After simplifying, we assume to the following result using lemma 4.3.

Theorem 4.6 The long run average cost per unit time for a multistate degenerative system under the bivariate replacement policy (T^+, N) with partial product process for the model outlined in section 3 under the A1 through A8 is provided by.

$$\mathscr{C}(T^+,N)=\frac{\left[\begin{array}{c}\left(\displaystyle\sum_{i=1}^{N-1}\frac{\mu}{\beta_0^{2^{i-1}}}[(1-b)F_N(T)+bF_i(T)]\right)c\\ +\left(\displaystyle\int_0^T udF_N(u)+\frac{F(T)}{\overline{F}(T)}\sum_{i=1}^{N-1}\frac{\lambda}{\alpha_0^{2^{i-1}}}[F_i(T)-F_N(T)]\right)r+c_p\tau+R\end{array}\right]}{\left[\begin{array}{c}\displaystyle\int_o^T udF_N(u)+\frac{F(T)}{\overline{F}(T)}\sum_{i=1}^{N-1}\frac{\lambda}{\alpha_0^{2^{i-1}}}[F_i(T)-F_N(T)]\\ +\displaystyle\sum_{i=1}^{N-1}\frac{\mu}{\beta_0^{2^{i-1}}}[(1-b)F_N(T)+bF_i(T)]+\tau\end{array}\right]} \tag{17}$$

The process used to determine the optimal policy $(T, N)^*$ is also employed to obtain the bivariate optimal replacement policy $(T^+, N)^*$ with partial product process.

4.4 The Bivariate Policy (U^-, N)

Under policy (U^-, N) states that the multistate degenerative system will be replaced at the failure point either upon the occurrence of the N-th failure, whichever comes first, or just before the total repair time exceeds U.

4.4.1 Virtual Repair Times

In the policy (U^-, N), an optimal policy might exist such that the system has to be replaced in the mid of a repair time. The question naturally arises whether it would not have been more beneficial to replace the system at the failure point itself as we might have saved on the repair cost. If fact, Stadje and Zuckerman (1992) have proved for their policy U that if Y_s's are new better then used in expectation, then there does exist an optimal replacement policy which does not replace in the middle of a repair period. Because since no additional cost is involved for replacing at failure in our policies, the strategy of not replacing system components in the middle of the operating cycle is economical.

4.4.2 The length of a cycle and its mean

The length of a cycle W under the bivariate replacement policy (U^-, N) with partial product process is

$$W = \left(\sum_{i=1}^{\eta} X_i + \sum_{i=o}^{v} Y_i\right) \chi(M_N > U) + \left(\sum_{i=1}^{N} X_i + \sum_{i=1}^{N-1} Y_i\right) \chi(M_N \leq U) + Z,$$

where the number of failures before the overall repair time above U is denotes by $\eta = 1, 2, \cdots, N-1$. and the number of repairs before the overall repair time above U is denoted by $v = 0, 1, 2, \cdots, N-1$.
If $M_i \leq U < M_{i+1}$ for $i = 1, 2, \cdots, N-1$, then $U - M_i$ will be the virtual repair time.

Lemma 4.7 The mean length of the cycle under policy (U^-, N) is

$$E(W) = \int_0^U u dG_N(u) + \frac{G(U)}{\overline{G}(U)} \sum_{i=1}^{N-1} \frac{\mu}{\beta_0^{2^{i-1}}} [G_i(U) - G_N(U)] + \sum_{i=1}^{N-1} \frac{\lambda}{\alpha_0^{2^{i-1}}} G_{i-1}(U) + \frac{\lambda}{\alpha_0^{2^{N-1}}} G_N(U) + \tau. \quad (18)$$

Proof. Consider

$$\begin{aligned} E(W) &= \left[\left(\sum_{i=1}^{\eta} X_i + \sum_{i=o}^{v} Y_i\right) \chi(M_N > U)\right] + E\left[\left(\sum_{i=1}^{N} X_i + \sum_{i=1}^{N-1} Y_i\right) \chi(M_N \leq U)\right] \\ &\quad + E(Z), \\ &= \int_0^U u dG_N(u) + \sum_{i=1}^{N} \frac{\lambda}{\alpha_0^{2^{N-1}}} G_N(U) + E\left[\sum_{i=1}^{\eta} X_{i\chi(M_N > U)}\right] \\ &\quad + E\left[\sum_{i=0}^{v} Y_{i\chi(M_N > U)}\right] + E(Z) \end{aligned}$$

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$$
\begin{aligned}
= & \int_0^U u dG_N(u) + \sum_{i=1}^{N} \frac{\lambda}{\alpha_0^{2^{N-1}}} G_N(U) + \sum_{i=1}^{N-1} E(X_i) P[M_{i-1} \leq U < M_N] \\
& + \sum_{i=0}^{N-1} E(Y_i \mid v) P[M_i \leq U < M_N] + E(Z) \\
= & \int_0^U u dG_N(u) + \sum_{i=1}^{N} \frac{\lambda}{\alpha_0^{2^{N-1}}} G_N(U) + \sum_{i=1}^{N-1} \frac{\lambda}{\alpha_0^{2^{i-1}}} [G_{i-1}(U) - G_N(U)] \\
& + \sum_{i=0}^{N-1} E(Y_i) E(v-1) [G_i(U) - G_N(U)] + \tau \\
= & \int_0^U u dG_N(u) + \frac{G(U)}{\overline{G}(U)} \sum_{i=1}^{N-1} \frac{\mu}{\beta_0^{2^{i-1}}} [G_i(U) - G_N(U)] \\
& + \sum_{i=1}^{N} \frac{\lambda}{\alpha_0^{2^{N-1}}} G_N(U) + \sum_{i=1}^{N-1} \frac{\lambda}{\alpha_0^{2^{i-1}}} [G_{i-1}(U) - G_N(U)] + \tau,
\end{aligned}
$$

as desired and this completes the proof. ■

4.4.3 The long-run average cost under the policy (U^-, N)

The long-run average cost per unit time under the multistate bivariate replacement policy (U^-, N) with partial product process.

$$
\begin{aligned}
\mathscr{C}(U^-, N) &= \frac{\text{the expected cost incurred in a cycle}}{\text{the expected length of a cycle}} \\
&= \frac{\left[\begin{array}{c} E\left\{ \left(c \sum_{i=0}^{v} Y_i - r \sum_{i=1}^{\eta} X_i \right) \chi(M_N > U) \right\} + c_p E(Z) \\ +E\left\{ \left(c \sum_{i=1}^{N-1} Y_i - r \sum_{i=1}^{N} X_i \right) \chi(M_N \leq U) \right\} + R \end{array} \right]}{E(W)}
\end{aligned}
$$

After simplifying, we assume to the following result using lamma 4.4.

Theorem 4.8 The long run average cost per unit time for a multistate degenerative system under the bivariate replacement policy (U^{-}, N) with partial product process for the model outlined in section 3 under the A1 through A8 is provided by.

$$\mathscr{C}(U^{-},N)=\frac{\left[\begin{array}{c}\left(\int_0^U udG_N(u)+\frac{G(U)}{\overline{G}(U)}\sum_{i=1}^{N-1}\frac{\mu}{\beta_0^{2^{i-1}}}[G_i(U)-G_N(U)]\right)c\\ -\left(\sum_{i=1}^{N-1}\frac{\lambda}{\alpha_0^{2^{i-1}}}G_{i-1}(U)+\frac{\lambda}{\alpha_0^{2^{N-1}}}G_N(U)\right)r+c_p\tau+R\end{array}\right]}{\left[\begin{array}{c}\sum_{i=1}^{N-1}\frac{\lambda}{\alpha_0^{2^{i-1}}}G_{i-1}(U)+\frac{\lambda}{\alpha_0^{2^{N-1}}}G_N(U)+\int_0^U udG_N(u)\\ +\frac{G(U)}{\overline{G}(U)}\sum_{i=1}^{N-1}\frac{\mu}{\beta_0^{2^{i-1}}}[G_i(U)-G_N(U)]+\tau\end{array}\right]} \tag{19}$$

5 Conclusion

By considering a repairable system for a monotone process model of a multi component multistate degenerative system, explicit expressions for the long-run average cost per unit time under a bivariate replacement policies $(T,N),(U,N),(T^{+},N)$ and (U^{-},N) with partial product process have been derived. Existence of optimal value of has been deduced. Numerical examples for some of the aforesaid bivariate replacement policies are given to illustrate the models and methodology developed in this paper.

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