



## Improved Independent Set Conditions for Fractional Factors

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### Abstract

A graph  $G$  is called a fractional  $(g, f, n', m)$ -critical deleted graph if after deleting any  $n'$  vertices from  $G$ , the resulting graph admits a fractional  $(g, f, m)$ -deleted graph. A graph  $G$  is called a fractional ID- $(g, f, m)$ -deleted if after deleting any independent set  $I$  from  $G$ , the resulting graph admits a fractional  $(g, f, m)$ -deleted graph. In this paper, we improve independent set conditions for a graph to be fractional  $(g, f, n', m)$ -critical deleted and fractional ID- $(g, f, m)$ -deleted. Furthermore, we present some examples to show the sharpness of given independent set bounds.

**Key words:** graph, fractional  $(g, f)$ -factor, fractional  $(g, f, n', m)$ -critical deleted graph, fractional ID- $(g, f, m)$ -deleted graph

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## 1. Introduction

In computer network, the fractional factor theory is used to test the feasibility of data transmission, and thus raises great attention from researchers. The purpose of this paper is to consider the data transmission problem from graph theory point of view, and only simple graphs are considered here. Let  $G = (V(G), E(G))$  be a graph with its vertex set  $V(G)$  and its edge set  $E(G)$ , and  $n = |V(G)|$  is the order of graph. Set  $e_G(S, T) = |\{e = uv : u \in S, v \in T\}|$  for any non-disjoint  $S, T \subset V(G)$ . The standard notations and terminologies in this paper can be referred to Bondy and Murty [1].

Let  $g$  and  $f$  be two integer-valued functions defined on vertex set satisfying  $0 \leq g(x) \leq f(x)$  for any  $x \in V(G)$ . A fractional  $(g, f)$ -factor is a real function

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$h : E \rightarrow [0, 1]$  meet  $g(x) \leq \sum_{e \in E(x)} h(e) \leq f(x)$  for each vertex  $x$ . A fractional  $f$ -factor is a special case of fractional  $(g, f)$ -factor if  $g(x) = f(x)$ ; a fractional  $[a, b]$ -factor if  $g(x) = a, f(x) = b$ ; a fractional  $k$ -factor if  $g(x) = f(x) = k$  for arbitrarily  $x \in V(G)$ , respectively. There are different names of graph in the different setting: fractional  $(g, f, m)$ -deleted graph (still has a fractional  $(g, f)$ -factor after deleting any  $m$  edges); fractional  $(g, f, n')$ -critical graph (still has a fractional  $(g, f)$ -factor after deleting any  $n'$  vertices); fractional  $(g, f, n', m)$ -critical deleted graph (still a fractional  $(g, f, m)$ -deleted graph after removing any  $n'$  vertices from  $G$ ). And, the corresponding fractional  $[a, b]$ -,  $f$ - and  $k$ - graphs can be well defined. In particular, if the deleted vertex set is an independent set, then it called fractional ID- $(g, f, m)$ -deleted graph (if  $G - I$  is a fractional  $(g, f, m)$ -deleted graph for any independent set  $I$  of  $G$ ), and the corresponding fractional ID- $(f, m)$ -deleted graph, fractional ID- $(a, b, m)$ -deleted graph and fractional ID- $(k, m)$ -deleted graph can be well defined. Some results on sufficient conditions for the existence of fractional factor can be referred to Gao and Wang [6] and [7], Wu et al. [8] and [9], and Zhou et al. [10], [11], [12], [13], [14] and [15].

Our main results are stated as follows which extend the previous results manifested in Gao et al. [3], [4] and [5]. The sharpness of the bounds will be presented in Section (3), and the detailed proofs will be presented in the next section.

**Theorem 1.1** Let  $G$  be a graph of order  $n$ . Let  $a, b, n', m, \Delta$  be five integers with  $i \geq 2, 2 \leq a \leq b - \Delta$  and  $n', m, \Delta \geq 0$ . Let  $g, f$  be two integer-valued functions defined on  $V(G)$  such that  $a \leq g(x) \leq f(x) - \Delta \leq b - \Delta$  for each  $x \in V(G)$ . If  $\delta(G) \geq \frac{b(b-\Delta)(i-1)}{a+\Delta} + n' + m, n > \frac{(a+b)(a+b+2m-1+(i-2)(b-\Delta))}{a+\Delta} + n'$ , and

$$\max\{d_G(x_1), d_G(x_2), \dots, d_G(x_i)\} \geq \frac{(b - \Delta)n + (a + \Delta)n'}{a + b}$$

for any independent subset  $\{x_1, x_2, \dots, x_i\}$  of  $V(G)$ , then  $G$  is a fractional  $(g, f, n', m)$ -critical deleted graph.

**Theorem 1.2** Let  $G$  be a graph of order  $n$ . Let  $a, b, n', m, \Delta$  be five integers with  $i \geq 2, 2 \leq a \leq b - \Delta$  and  $n', m, \Delta \geq 0$ . Let  $g, f$  be two integer-valued functions defined on  $V(G)$  such that  $a \leq g(x) \leq f(x) - \Delta \leq b - \Delta$  for each  $x \in V(G)$ . If  $\delta(G) \geq \frac{b(b - \Delta)(i - 1)}{a + \Delta} + n' + m, n > \frac{(a + b)(i(a + b) + 2m - 2)}{a + \Delta} + n'$ , and

$$|N_G(x_1) \cup N_G(x_2) \cup \dots \cup N_G(x_i)| \geq \frac{(b - \Delta)n + (a + \Delta)n'}{a + b}$$

for any independent subset  $\{x_1, x_2, \dots, x_i\}$  of  $V(G)$ , then  $G$  is a fractional  $(g, f, n', m)$ -critical deleted graph.

Set  $n' = 0$  in Theorem (1.1) and Theorem (1.2), then the above two conclusions become Theorem 1 and Theorem 2 in Gao et al. [3] which reveal the independent set conditions for fractional  $(g, f, m)$ -deleted graphs.

**Corollary 1.3** Let  $G$  be a graph of order  $n$ . Let  $a, b, m, \Delta$  be five integers with  $i \geq 2$ ,  $2 \leq a \leq b - \Delta$  and  $m, \Delta \geq 0$ . Let  $g, f$  be two integer-valued functions defined on  $V(G)$  such that  $a \leq g(x) \leq f(x) - \Delta \leq b - \Delta$  for each  $x \in V(G)$ . If  $\delta(G) \geq \frac{b(b - \Delta)(i - 1)}{a + \Delta} + m$ ,  $n > \frac{(a + b)(a + b + 2m - 1 + (i - 2)(b - \Delta))}{a + \Delta}$ , and

$$\max\{d_G(x_1), d_G(x_2), \dots, d_G(x_i)\} \geq \frac{(b - \Delta)n}{a + b}$$

for any independent subset  $\{x_1, x_2, \dots, x_i\}$  of  $V(G)$ , then  $G$  is a fractional  $(g, f, m)$ -deleted graph.

**Corollary 1.4** Let  $G$  be a graph of order  $n$ . Let  $a, b, m, \Delta$  be five integers with  $i \geq 2$ ,  $2 \leq a \leq b - \Delta$  and  $m, \Delta \geq 0$ . Let  $g, f$  be two integer-valued functions defined on  $V(G)$  such that  $a \leq g(x) \leq f(x) - \Delta \leq b - \Delta$  for each  $x \in V(G)$ . If  $\delta(G) \geq \frac{b(b - \Delta)(i - 1)}{a + \Delta} + m$ ,  $n > \frac{(a + b)(i(a + b) + 2m - 2)}{a + \Delta}$ , and

$$|N_G(x_1) \cup N_G(x_2) \cup \dots \cup N_G(x_i)| \geq \frac{(b - \Delta)n}{a + b}$$

for any independent subset  $\{x_1, x_2, \dots, x_i\}$  of  $V(G)$ , then  $G$  is a fractional  $(g, f, m)$ -deleted graph.

Taking  $m = 0$  in Theorem (1.1) and Theorem (1.2), we obtain the corresponding independent set conditions for fractional  $(g, f, n')$ -critical graphs. Furthermore, by setting  $g(x) = a$  and  $f(x) = b$ , the corresponding independent set conditions for fractional  $(a, b, n', m)$ -critical deleted graphs are analyzed. We don't list these corollaries one by one. Moreover, we don't consider the setting in  $g(x) = f(x)$  for each  $x \in V(G)$  due to the corresponding conclusions on  $\Delta = 0$  are already manifested in other published papers.

When it comes to fractional fractional ID- $(g, f, m)$ -deleted graph setting, we

determine the following two theorems on independent set degree condition and independent set neighborhood union condition, respectively.

**Theorem 1.5** Let  $G$  be a graph of order  $n$ . Let  $a, b, m, \Delta$  be five integers with  $i \geq 2$ ,  $2 \leq a \leq b - \Delta$  and  $m, \Delta \geq 0$ . Let  $g, f$  be two integer-valued functions defined on  $V(G)$  such that  $a \leq g(x) \leq f(x) - \Delta \leq b - \Delta$  for each  $x \in V(G)$ . If  $\delta(G) \geq \frac{(a + \Delta)n}{2a + \Delta + b} + \frac{b(b - \Delta)(i - 1)}{a + \Delta} + m$ ,  $n > \frac{(2a + b)(a + b + 2m - 1 + (i - 2)(b - \Delta))}{a + \Delta}$ ,

and

$$\max\{d_G(x_1), d_G(x_2), \dots, d_G(x_i)\} \geq \frac{(a + b)n}{2a + \Delta + b}$$

for any independent subset  $\{x_1, x_2, \dots, x_i\}$  of  $V(G)$ , then  $G$  is a fractional ID- $(g, f, m)$ -deleted graph.

**Theorem 1.6** Let  $G$  be a graph of order  $n$ , and let  $a, b, i, m, \Delta$  be five nonnegative integers with  $i \geq 2$ ,  $2 \leq a \leq b - \Delta$ ,  $n > \frac{(2a + b)(i(a + b) + 2m - 2)}{a + \Delta}$  and  $\delta(G) \geq$

$\frac{(a + \Delta)n}{2a + \Delta + b} + \frac{b(b - \Delta)(i - 1)}{a + \Delta} + m$ . Let  $g, f$  be two integer-valued functions defined on  $V(G)$  such that  $a \leq g(x) \leq f(x) - \Delta \leq b - \Delta$  for each  $x \in V(G)$ . If  $G$  satisfies

$$|N_G(x_1) \cup N_G(x_2) \cup \dots \cup N_G(x_i)| \geq \frac{(a + b)n}{2a + \Delta + b}$$

for any independent subset  $\{x_1, x_2, \dots, x_i\}$  of  $V(G)$ , then  $G$  is a fractional ID- $(g, f, m)$ -deleted graph.

The corresponding sufficient conditions for fractional ID- $(g, f)$ -factor-critical graphs can be yield by setting  $m = 0$  in Theorem (1.5) and Theorem (1.6), respectively. In particular, in the case  $g(x) = a$  and  $f(x) = b$  for each  $x \in V(G)$ , the independent set conditions for fractional ID- $(a, b, m)$ -deleted graph can be derived. Other setting regard  $\Delta = 0$  are not considered in this paper.

Proof of our main results mainly depended on the following lemma which present the necessary and sufficient condition of a graph to be fractional  $(g, f, n', m)$ -critical deleted.

**Lemma 1.7** (Gao [2]) Let  $G$  be a graph,  $g, f$  be two integer-valued functions defined on  $V(G)$  such that  $g(x) \leq f(x)$  for each  $x \in V(G)$ . Let  $n', m$  be two non-negative

integers. Then  $G$  is fractional  $(g, f, n', m)$ -critical deleted graph if and only if

$$f(S) - g(T) + d_{G-S}(T) \geq \max_{U \subseteq S, |U|=n', H \subseteq E(G-U), |H|=m} \{f(U) + \sum_{x \in T} d_H(x) - e_H(T, S)\}$$

for all disjoint subsets  $S, T$  of  $V(G)$  with  $|S| \geq n'$ .

By taking  $n' = 0$  in Lemma (1.7), we deduce the necessary and sufficient condition of a graph to be fractional  $(g, f, m)$ -deleted which will be used to explain the sharpness of Theorem (1.5) and Theorem (1.6).

**Lemma 1.8** Let  $G$  be a graph,  $g, f$  be two integer-valued functions defined on  $V(G)$  such that  $g(x) \leq f(x)$  for each  $x \in V(G)$ . Let  $m$  be two non-negative integers. Then  $G$  is a fractional  $(g, f, m)$ -deleted graph if and only if

$$f(S) - g(T) + d_{G-S}(T) \geq \max_{H \subseteq E(G), |H|=m} \left\{ \sum_{x \in T} d_H(x) - e_H(T, S) \right\}$$

for all disjoint subsets  $S, T$  of  $V(G)$ .

## 2. Proof of Main Results

The purpose of this section is to present the detailed proofing procedures of four results manifested in last section. The tricks used in this section are mainly followed by Gao et al. [3], [4] and [5].

### Proof of Theorem (1.1)

Assume that  $G$  satisfies the conditions of the Theorem (1.1) but it's not a fractional  $(g, f, n', m)$ -critical deleted graph. By Lemma (1.7) and  $\sum_{x \in T} d_H(x) - e_H(T, S) \leq 2m$ , there exist disjoint subsets  $S$  and  $T$  of  $V(G)$  satisfying

$$(a + \Delta)(|S| - n') + d_{G-S}(T) - (b - \Delta)|T| \leq f(S - U) + d_{G-S}(T) - g(T) \leq 2m - 1, \quad (1)$$

where  $|S| \geq n' = |U|$ . We select subsets  $S$  and  $T$  such that  $|T|$  is minimal. Obviously,  $T \neq \emptyset$  and  $d_{G-S}(x) \leq g(x) - 1 \leq b - \Delta - 1$  for any  $x \in T$ .

Let  $d_1 = \min\{d_{G-S}(x) | x \in T\}$  and select  $x_1 \in T$  with  $d_{G-S}(x_1) = d_1$ . If  $z \geq 2$  and  $T \setminus (\cup_{j=1}^{z-1} N_T[x_j]) \neq \emptyset$ , let

$$d_z = \min\{d_{G-S}(x) | x \in T \setminus (\cup_{j=1}^{z-1} N_T[x_j])\}$$

and select  $x_z \in T \setminus (\cup_{j=1}^{z-1} N_T[x_j])$  with  $d_{G-S}(x_z) = d_z$ . Thus, it generates a sequence with  $0 \leq d_1 \leq d_2 \leq \dots \leq d_\pi \leq g(x) - 1 \leq b - \Delta - 1$  and an independent set  $\{x_1, x_2, \dots, x_\pi\} \subseteq T$ .

Proof: Suppose  $|T| \leq (i-1)b$ . Then  $|S| + d_1 \geq d_G(x_1) \geq \delta(G) \geq \frac{b(b-\Delta)(i-1)}{a+\Delta} + n' + m$ . According to (1) and  $0 \leq d_1 \leq b - \Delta - 1$ , we get

$$\begin{aligned} 2m - 1 &\geq (a + \Delta)(|S| - n') + d_{G-S}(T) - (b - \Delta)|T| \\ &\geq (a + \Delta)(|S| - n') + d_1|T| - (b - \Delta)|T| \\ &= (a + \Delta)(|S| - n') + (d_1 + \Delta - b)|T| \\ &\geq (a + \Delta)\left(\frac{b(b-\Delta)(i-1)}{a+\Delta} - d_1 + m\right) + (d_1 + \Delta - b)(i-1)b \\ &\geq 2m. \end{aligned}$$

This produces a contradiction.

Since  $d_{G-S}(x) \leq b - \Delta - 1$  and  $|T| \geq (i-1)b + 1$ , we obtain  $\pi \geq i$ . Hence, we can select an independent set  $\{x_1, x_2, \dots, x_i\} \subseteq T$ .

In light of independent set neighborhood union condition described in the theorem, we infer

$$\frac{(b-\Delta)n + (a+\Delta)n'}{a+b} \leq \max\{d_G(x_1), d_G(x_2), \dots, d_G(x_i)\} \leq |S| + d_i$$

and

$$|S| \geq \frac{(b-\Delta)n + (a+\Delta)n'}{a+b} - d_i. \quad (2)$$

Since

$$|N_T[x_j]| - |N_T[x_j] \cap (\cup_{z=1}^{j-1} N_T[x_z])| \geq 1, j = 2, 3, \dots, i-1$$

and

$$|\cup_{z=1}^j N_T[x_z]| \leq \sum_{z=1}^j |N_T[x_z]| \leq \sum_{z=1}^j (d_{G-S}(x_z) + 1) = \sum_{z=1}^j (d_z + 1), j = 1, 2, \dots, i,$$

we deduce  $f(S - U) + d_{G-S}(T) - g(T)$

$$\begin{aligned} &\geq (a + \Delta)(|S| - n') - (b - \Delta)|T| + d_1|N_T[x_1]| \\ &\quad + d_2(|N_T[x_2]| - |N_T[x_2] \cap N_T[x_1]|) + \dots \\ &\quad + d_{i-1}(|N_T[x_{i-1}]| - |N_T[x_{i-1}] \cap (\cup_{j=1}^{i-2} N_T[x_j])|) + d_i(|T| - |(\cup_{j=1}^{i-1} N_T[x_j])|) \end{aligned}$$

$$\begin{aligned}
 &\geq (a + \Delta)(|S| - n') + (d_1 - d_i)|N_T[x_1]| + \sum_{j=2}^{i-1} d_j + (d_i + \Delta - b)|T| - d_i \sum_{j=2}^{i-1} |N_T[x_j]| \\
 &= (a + \Delta)(|S| - n') + (d_1 - d_i)(d_1 + 1) + \sum_{j=2}^{i-1} d_j + (d_i + \Delta - b)|T| - d_i \sum_{j=2}^{i-1} (d_j + 1) \\
 &= (a + \Delta)(|S| - n') + d_1^2 + \sum_{j=1}^{i-1} d_j + (d_i + \Delta - b)|T| - d_i \sum_{j=1}^{i-1} (d_j + 1),
 \end{aligned}$$

which implies

$$\begin{aligned}
 &(n - |S| - |T|)(b - \Delta - d_i) \\
 &\geq f(S - U) + d_{G-S}(T) - g(T) - 2m + 1 \\
 &\geq (a + \Delta)(|S| - n') + d_1^2 + \sum_{j=1}^{i-1} d_j + (d_i + \Delta - b)|T| - d_i \sum_{j=1}^{i-1} (d_j + 1) - 2m + 1.
 \end{aligned}$$

Equivalently,

$$0 \leq n(b - \Delta - d_i) - (a + b - d_i)|S| + d_i \sum_{j=1}^{i-1} d_j - \sum_{j=1}^{i-1} d_j + d_i(i - 1) - d_1^2 + (a + \Delta)n' + 2m - 1. \quad (3)$$

By means of (2), (3),  $d_1 \leq d_2 \leq \dots \leq d_i \leq b - \Delta - 1$  and

$$n > \frac{(a + b)(a + b + 2m - 1 + (i - 2)(b - \Delta))}{a + \Delta} + n', \text{ we have}$$

$$\begin{aligned}
 0 &\leq n(b - \Delta - d_i) - (a + b - d_i) \left( \frac{(b - \Delta)n + (a + \Delta)n'}{a + b} - d_i \right) + d_i \sum_{j=1}^{i-1} d_j - \sum_{j=1}^{i-1} d_j \\
 &\quad + d_i(i - 1) - d_1^2 + (a + \Delta)n' + 2m - 1 \\
 &= -\frac{(a + \Delta)n}{a + b} d_i + (a + b)d_i - d_i^2 + d_i \sum_{j=1}^{i-1} d_j - \sum_{j=1}^{i-1} d_j + d_i(i - 1) - d_1^2 + \frac{(a + \Delta)d_i}{a + b} n' + 2m - 1
 \end{aligned}$$

$$\begin{aligned}
 &= -\frac{(a+\Delta)n}{a+b}d_i + (d_id_1 - d_1 - d_1^2) + (d_i - 1)\sum_{j=2}^{i-1} d_j \\
 &\quad + d_i(a+b+i-1) - d_i^2 + \frac{(a+\Delta)d_i}{a+b}n' + 2m - 1 \\
 &\leq -\frac{(a+\Delta)n}{a+b}d_i + (d_i\frac{d_i-1}{2} - \frac{d_i-1}{2} - (\frac{d_i-1}{2})^2) + (d_i - 1)\sum_{j=2}^{i-1} d_j \\
 &\quad + d_i(a+b+i-1) - d_i^2 + \frac{(a+\Delta)d_i}{a+b}n' + 2m - 1 \\
 &= -\frac{(a+\Delta)n}{a+b}d_i + (i - \frac{11}{4})d_i^2 + d_i(a+b + \frac{1}{2}) - \frac{3}{4} + \frac{(a+\Delta)d_i}{a+b}n' + 2m.
 \end{aligned}$$

If  $d_i > 0$ , then  $0 < -\frac{(a+\Delta)n}{a+b}d_i + (i - \frac{11}{4})d_i^2 + d_i(a+b + \frac{1}{2}) - \frac{3}{4} + \frac{(a+\Delta)d_i}{a+b}n' + 2m \leq 0$   
 since  $n > \frac{(a+b)(a+b+2m+(a+\Delta)n'-1+(i-2)(b-\Delta))}{a+\Delta}$  and  $2m(1-d_i) \leq 0$ ,  
 a contradiction.

If  $d_i = 0$ , then  $d_1 = \dots = d_i = 0$ . In view of (2), we get we get  $|S| \geq \frac{(b-\Delta)n + (a+\Delta)n'}{a+b}$  and  $|T| \leq n - |S| \leq \frac{(a+\Delta)n - (a+\Delta)n'}{a+b}$ . By means of  $d_{G-S}(T) \geq \sum_{x \in T} d_H(x) - e_G(T, S)$ , we yield

$$\begin{aligned}
 &f(S-U) + d_{G-S}(T) - g(T) - (\sum_{x \in T} d_H(x) - e_G(T, S)) \\
 &\geq (a+\Delta) \cdot (\frac{(b-\Delta)n + (a+\Delta)n'}{a+b} - n') - (b-\Delta) \cdot \frac{(a+\Delta)n - (a+\Delta)n'}{a+b} \\
 &\quad + (d_{G-S}(T) - \sum_{x \in T} d_H(x) + e_G(T, S)) \\
 &\geq 0,
 \end{aligned}$$

is also a contradiction.

Therefore, the desired theorem is proved.

### **Proof of Theorem (1.2)**

On the contrary, assume that  $G$  satisfies the conditions of the Theorem (1.2), but it's not a fractional  $(g, f, n', m)$ -critical deleted graph. By Lemma (1.7) and



$\sum_{x \in T} d_H(x) - e_H(T, S) \leq 2m$ , there exist disjoint subsets  $S$  and  $T$  of  $V(G)$  satisfying

$$(a + \Delta)(|S| - n') + d_{G-S}(T) - (b - \Delta)|T| \leq f(S - U) + d_{G-S}(T) - g(T) \leq 2m - 1, \quad (4)$$

where  $|S| \geq n'$ . We select subsets  $S$  and  $T$  such that  $|T|$  is minimal. Obviously,  $T \neq \emptyset$  and  $d_{G-S}(x) \leq g(x) - 1 \leq b - \Delta - 1$  for any  $x \in T$ .

Let  $d_1 = \min\{d_{G-S}(x) | x \in T\}$  and select  $x_1 \in T$  with  $d_{G-S}(x_1) = d_1$ . If  $z \geq 2$  and  $T \setminus (\cup_{j=1}^{z-1} N_T[x_j]) \neq \emptyset$ , let

$$d_z = \min\{d_{G-S}(x) | x \in T \setminus (\cup_{j=1}^{z-1} N_T[x_j])\}$$

and select  $x_z \in T \setminus (\cup_{j=1}^{z-1} N_T[x_j])$  with  $d_{G-S}(x_z) = d_z$ . Thus, it generates a sequence with  $0 \leq d_1 \leq d_2 \leq \dots \leq d_\pi \leq g(x) - 1 \leq b - \Delta - 1$  and an independent set  $\{x_1, x_2, \dots, x_\pi\} \subseteq T$ . Using the trick as depicted in the proofing of Theorem (1.1), we infer  $|T| \geq (i - 1)b + 1$ . Since  $d_{G-S}(x) \leq b - \Delta - 1$  and  $|T| \geq (i - 1)b + 1$ , we obtain  $\pi \geq i$ . Hence, we can select an independent set  $\{x_1, x_2, \dots, x_i\} \subseteq T$ .

In light of independent set neighborhood union condition described in the theorem, we infer

$$\frac{(b - \Delta)n + (a + \Delta)n'}{a + b} \leq |N_G(x_1) \cup N_G(x_2) \cup \dots \cup N_G(x_i)| \leq |S| + \sum_{j=1}^i d_j$$

and

$$|S| \geq \frac{(b - \Delta)n + (a + \Delta)n'}{a + b} - \sum_{j=1}^i d_j. \quad (5)$$

Using the same fashion, we have

$$0 \leq n(b - \Delta - d_i) - (a + b - d_i)|S| + d_i \sum_{j=1}^{i-1} d_j - \sum_{j=1}^{i-1} d_j + d_i(i - 1) - d_1^2 + (a + \Delta)n' + 2m - 1. \quad (6)$$

By means of (5), (6),  $d_1 \leq d_2 \leq \dots \leq d_i \leq b - \Delta - 1$  and  $n > \frac{(a+b)(i(a+b)+2m-2)}{a+\Delta} + n'$ , we have

$$0 \leq n(b - \Delta - d_i) - (a + b - d_i) \left( \frac{(b - \Delta)n + (a + \Delta)n'}{a + b} - \sum_{j=1}^i d_j \right) + d_i \sum_{j=1}^{i-1} d_j - \sum_{j=1}^{i-1} d_j + d_i(i - 1) - d_1^2 + (a + \Delta)n' + 2m - 1$$

$$\begin{aligned}
 &= -\frac{(a+\Delta)n}{a+b}d_i + (a+b)\sum_{j=1}^i d_j - d_i \sum_{j=1}^i d_j + d_i \sum_{j=1}^{i-1} d_j - \sum_{j=1}^{i-1} d_j \\
 &\quad + d_i(i-1) - d_1^2 + d_i \frac{(a+\Delta)n'}{a+b} + 2m-1 \\
 &= -\frac{(a+\Delta)n}{a+b}d_i + ((a+b-1)d_1 - d_1^2) + (a+b-1)\sum_{j=2}^{i-1} d_j \\
 &\quad + d_i(a+b+i-1) - d_i^2 + d_i \frac{(a+\Delta)n'}{a+b} + 2m-1 \\
 &\leq -\frac{(a+\Delta)n}{a+b}d_i + (a+b-1)d_i + (a+b-1)\sum_{j=2}^{i-1} d_i \\
 &\quad + d_i(a+b+i-1) - d_i^2 + d_i \frac{(a+\Delta)n'}{a+b} + 2m-1 \\
 &= -\frac{(a+\Delta)n}{a+b}d_i + i(a+b)d_i - d_i^2 + d_i \frac{(a+\Delta)n'}{a+b} + 2m-1
 \end{aligned}$$

If  $d_i > 0$ , then  $0 < -\frac{(a+\Delta)n}{a+b}d_i + i(a+b)d_i - d_i^2 + d_i \frac{(a+\Delta)n'}{a+b} + 2m-1 \leq 0$  since  $n > \frac{(a+b)(i(a+b)+2m-2)}{a+\Delta} + n'$  and  $2m(1-d_i) \leq 0$ , a contradiction.

If  $d_i = 0$ , then  $d_1 = \dots = d_i = 0$ . In view of (5), we get  $|S| \geq \frac{(b-\Delta)n + (a+\Delta)n'}{a+b}$  and  $|T| \leq n - |S| \leq \frac{(a+\Delta)n - (a+\Delta)n'}{a+b}$ . By means of  $d_{G-S}(T) \geq \sum_{x \in T} d_H(x) - e_G(T, S)$ , we yield

$$\begin{aligned}
 &f(S-U) + d_{G-S}(T) - g(T) - \left(\sum_{x \in T} d_H(x) - e_G(T, S)\right) \\
 &\geq (a+\Delta) \cdot \left(\frac{(b-\Delta)n + (a+\Delta)n'}{a+b} - n'\right) - (b-\Delta) \cdot \frac{(a+\Delta)n - (a+\Delta)n'}{a+b} \\
 &\quad + (d_{G-S}(T) - \sum_{x \in T} d_H(x) + e_G(T, S)) \\
 &\geq 0,
 \end{aligned}$$

is also a contradiction.

Therefore, the desired theorem is proved.

**Proof of Theorem (1.5) and (1.6)**

Here we only present the proof of Theorem (1.6), and the proof of Theorem (1.5) can be done by using the similar techniques.

Set  $G' = G - I$  for any independent set  $I$ . If  $|I| = 1$ , then  
 $|V(G')| > \frac{(2a+b)(i(a+b)+2m-2)}{a+\Delta} - 1 > \frac{(a+b)(i(a+b)+2m-2)}{a+\Delta}$ . It's not hard to verify that  $\delta(G') \geq \frac{b(b-\Delta)(i-1)}{a+\Delta} + m$  and  $|N_{G'}(x_1) \cup N_{G'}(x_2) \cup \dots \cup N_{G'}(x_i)| \geq \frac{b|V(G')|}{a+b} = \frac{b(n-1)}{a+b}$  for any independent subset  $\{x_1, x_2, \dots, x_i\}$  of  $G'$ . Thus, the result holds from Corollary (1.4).

We now discuss the case  $|I| \geq 2$ . By means of independent set neighborhood union condition, we derive  $|V(G')| \geq \frac{(a+b)n}{2a+b+\Delta} > \frac{(a+b)(i(a+b)+2m-2)}{a+\Delta}$ . If  $|N_{G'}(x_1) \cup N_{G'}(x_2) \cup \dots \cup N_{G'}(x_i)| < \frac{(b-\Delta)|V(G')|}{a+b}$  for some independent subset  $\{x_1, x_2, \dots, x_i\}$  of  $V(G')$ , then  $\frac{(a+b)(|V(G')| + |I|)}{2a+b+\Delta} \leq |N_G(x_1) \cup N_G(x_2) \cup \dots \cup N_G(x_i)| < \frac{(b-\Delta)|V(G')|}{a+b} + |I|$ , i.e.,  $|V(G')| < \frac{a+b}{a+\Delta}|I| \leq \frac{a+b}{a+\Delta} \frac{(a+\Delta)n}{2a+b+\Delta} = \frac{(a+b)n}{2a+b+\Delta}$ . It contradicts  $|N_G(x_1) \cup N_G(x_2) \cup \dots \cup N_G(x_i)| \geq \frac{(a+b)n}{2a+b+\Delta}$  for any independent subset  $\{x_1, x_2, \dots, x_i\}$  of  $V(G)$  and  $|I| \geq 2$ . Therefore, we have  $|N_{G'}(x_1) \cup N_{G'}(x_2) \cup \dots \cup N_{G'}(x_i)| \geq \frac{(b-\Delta)|V(G')|}{a+b}$  for all independent subset  $\{x_1, x_2, \dots, x_i\}$  in  $G'$ . Furthermore, we infer  $\delta(G') \geq \frac{b(b-\Delta)(i-1)}{a} + m$  by  $|I| \leq \frac{(a+\Delta)n}{2a+b+\Delta}$  and  $\delta(G) \geq \frac{(a+\Delta)n}{2a+b+\Delta} + \frac{b(b-\Delta)(i-1)}{a} + m$ . At last, the result follows from Corollary (1.4). In all, Theorem (1.6) is hold.

### 3. Sharpness

The aim of this section is to present that the independent set results in Theorem (1.1)-(1.6) are tight.

Theorem (1.1) and Theorem (1.2) are best possible, in some extent, on the conditions. Actually, we can construct some graphs such that the independent set degree condition in Theorem (1.1) can't be replaced by  $\max\{d_G(x_1), d_G(x_2), \dots, d_G(x_i)\}$

$\geq \frac{(b - \Delta)n + (a + \Delta)n'}{a + b} - 1$ , and the independent set neighborhood union condition in Theorem (1.2) can't be replaced by  $|N_G(x_1) \cup N_G(x_2) \cup \dots \cup N_G(x_i)| \geq \frac{(b - \Delta)n + (a + \Delta)n'}{a + b} - 1$ .

Let  $b = a + \Delta$ ,  $G_1 = K_{iat+n'}$  be a complete graph,  $G_2 = (ibt + 1)K_1$  be a graph consisting of  $ibt + 1$  isolated vertices, and  $G = G_1 \vee G_2$ , where  $t$  is sufficiently large. Then  $n = |G_1| + |G_2| = i(a + b)t + n' + 1$ , and for any independent set  $\{x_1, x_2, \dots, x_i\} \subseteq V(G_2)$ , we get

$$\frac{(b - \Delta)n + (a + \Delta)n'}{a + b} > \max\{d_G(x_1), d_G(x_2), \dots, d_G(x_i)\} = iat + n' > \frac{(b - \Delta)n + (a + \Delta)n'}{a + b} - 1,$$

$$\frac{(b - \Delta)n + (a + \Delta)n'}{a + b} > |N_G(x_1) \cup N_G(x_2) \cup \dots \cup N_G(x_i)| = iat + n' > \frac{(b - \Delta)n + (a + \Delta)n'}{a + b} - 1.$$

Let  $S = V(G_1)$ ,  $T = V(G_2)$ ,  $g(x) = a$  and  $f(x) = b = a + \Delta$  for any  $x \in V(G)$ . Then  $f(S \setminus U) - g(T) + d_{G-S}(T) - (\sum_{x \in T} d_H(x) - e_H(T, S)) = biat - a(ibt + 1) = -a < 0$  for any  $U \subseteq S$  and  $|U| = n'$ . Hence,  $G$  is not a fractional  $(g, f, n', m)$ -critical deleted graph according to Lemma (1.7).

Next, we show that Theorem (1.5) and Theorem (1.6) are best possible, i.e., independent set degree condition can't be replaced by  $\max\{d_G(x_1), d_G(x_2), \dots, d_G(x_i)\} \geq \frac{(a + b)n}{2a + \Delta + b} - 1$ , and independent set neighborhood union condition can't be replaced by  $|N_G(x_1) \cup N_G(x_2) \cup \dots \cup N_G(x_i)| \geq \frac{(a + b)n}{2a + \Delta + b} - 1$ .

Considering a graph  $G = (bt + 1)K_1 \vee K_{at} \vee (bt + 1)K_1$ , where  $t$  is a sufficiently large positive integer. Clearly,  $n = (2b + a)t + 2$ . Let  $g(x) = a$  and  $f(x) = b = a + \Delta$  for all  $x \in V(G)$ . We have

$$\frac{(a + b)n}{2a + b + \Delta} > \max\{d_G(x_1), d_G(x_2), \dots, d_G(x_i)\} = (a + b)t + 1 > \frac{(a + b)n}{2a + b + \Delta} - 1$$

$$\frac{(a + b)n}{2a + b + \Delta} > |N_G(x_1) \cup N_G(x_2) \cup \dots \cup N_G(x_i)| = (a + b)t + 1 > \frac{(a + b)n}{2a + b + \Delta} - 1$$

for any independent subset  $\{x_1, x_2, \dots, x_i\}$  of  $V(G)$ .

Let  $I = (bt + 1)K_1$ . For  $G' = K_{at} \vee (bt + 1)K_1$ , let  $S = K_{at}$  and  $T = (bt + 1)K_1$ .

Then we have  $\sum_{x \in T} d_H(x) - e_H(T, S) = 0$  for any subset  $H$  of  $E(G')$  with  $m$  edges. Therefore,

$$\begin{aligned} f(S) - g(T) + d_{G'-S}(T) - \left( \sum_{x \in T} d_H(x) - e_H(T, S) \right) &= b(at) - a(bt + 1) \\ &= -a. \end{aligned}$$

Hence, by Lemma (1.8),  $G'$  is not a fractional  $(g, f, m)$ -deleted graph. In conclusion,  $G$  is not a fractional ID- $(g, f, m)$ -deleted graph.

#### 4. More remarks

If we allow  $a = 1$  in theorems, it is found that the minimal degree condition should be strengthened in order to meet the same independent set conditions. Specifically, using the proofing processors, we can obtain the following result.

**Theorem 4.1** Let  $G$  be a graph of order  $n$ . Let  $a, b, n', m, \Delta$  be five integers with  $i \geq 2$ ,  $1 \leq a \leq b - \Delta$  and  $n', m, \Delta \geq 0$ . Let  $g, f$  be two integer-valued functions defined on  $V(G)$  such that  $a \leq g(x) \leq f(x) - \Delta \leq b - \Delta$  for each  $x \in V(G)$ . If  $\delta(G) \geq \frac{b(b - \Delta)(i - 1)}{a + \Delta} + n' + 2m$ ,  $n > \frac{(a + b)(a + b + 2m - 1 + (i - 2)(b - \Delta))}{a + \Delta} + n'$ , and

$$\max\{d_G(x_1), d_G(x_2), \dots, d_G(x_i)\} \geq \frac{(b - \Delta)n + (a + \Delta)n'}{a + b}$$

for any independent subset  $\{x_1, x_2, \dots, x_i\}$  of  $V(G)$ , then  $G$  is a fractional  $(g, f, n', m)$ -critical deleted graph.

**Theorem 4.2** Let  $G$  be a graph of order  $n$ . Let  $a, b, n', m, \Delta$  be five integers with  $i \geq 2$ ,  $1 \leq a \leq b - \Delta$  and  $n', m, \Delta \geq 0$ . Let  $g, f$  be two integer-valued functions defined on  $V(G)$  such that  $a \leq g(x) \leq f(x) - \Delta \leq b - \Delta$  for each  $x \in V(G)$ . If  $\delta(G) \geq \frac{b(b - \Delta)(i - 1)}{a + \Delta} + n' + 2m$ ,  $n > \frac{(a + b)(i(a + b) + 2m - 2) + (a + \Delta)n'}{a + \Delta}$ , and

$$|N_G(x_1) \cup N_G(x_2) \cup \dots \cup N_G(x_i)| \geq \frac{(b - \Delta)n + (a + \Delta)n'}{a + b}$$

for any independent subset  $\{x_1, x_2, \dots, x_i\}$  of  $V(G)$ , then  $G$  is a fractional  $(g, f, n', m)$ -critical deleted graph.

**Theorem 4.3** Let  $G$  be a graph of order  $n$ . Let  $a, b, m, \Delta$  be five integers with  $i \geq 2$ ,  $1 \leq a \leq b - \Delta$  and  $m, \Delta \geq 0$ . Let  $g, f$  be two integer-valued functions defined on  $V(G)$  such that  $a \leq g(x) \leq f(x) - \Delta \leq b - \Delta$  for each  $x \in V(G)$ . If  $\delta(G) \geq \frac{(a + \Delta)n}{2a + \Delta + b} + \frac{b(b - \Delta)(i - 1)}{a + \Delta} + 2m$ ,  $n > \frac{(2a + b)(a + b + 2m - 1 + (i - 2)(b - \Delta))}{a + \Delta}$ , and

$$\max\{d_G(x_1), d_G(x_2), \dots, d_G(x_i)\} \geq \frac{(a + b)n}{2a + \Delta + b}$$

for any independent subset  $\{x_1, x_2, \dots, x_i\}$  of  $V(G)$ , then  $G$  is a fractional ID- $(g, f, m)$ -deleted graph.

**Theorem 4.4** Let  $G$  be a graph of order  $n$ , and let  $a, b, i, m, \Delta$  be five nonnegative integers with  $i \geq 2$ ,  $1 \leq a \leq b - \Delta$ ,  $n > \frac{(2a + b)(i(a + b) + 2m - 2)}{a + \Delta}$  and  $\delta(G) \geq \frac{(a + \Delta)n}{2a + \Delta + b} + \frac{b(b - \Delta)(i - 1)}{a + \Delta} + 2m$ . Let  $g, f$  be two integer-valued functions defined on  $V(G)$  such that  $a \leq g(x) \leq f(x) - \Delta \leq b - \Delta$  for each  $x \in V(G)$ . If  $G$  satisfies

$$|N_G(x_1) \cup N_G(x_2) \cup \dots \cup N_G(x_i)| \geq \frac{(a + b)n}{2a + \Delta + b}$$

for any independent subset  $\{x_1, x_2, \dots, x_i\}$  of  $V(G)$ , then  $G$  is a fractional ID- $(g, f, m)$ -deleted graph.

Again, the examples presented in Section (3) also show that the independent set degree and neighborhood union conditions are best possible.

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