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A Mathematical Framework for Social Network Analysis Using Graph Theory

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Abstract

The rapid growth of online social networking platforms such as Facebook, Twitter, and WhatsApp has created complex interaction structures that cannot be fully understood using traditional statistical methods alone. This study aims to model and interpret the structural properties of social networks using graph theory as a mathematical framework. In this representation, users are modeled as vertices and interactions such as friendships, follows, and message exchanges are represented as edges. The methodology involves constructing directed and undirected graphs from real-world inspired data and analyzing them using graph-theoretic concepts such as degree distribution, connectivity, paths, cycles, bridges, cut-vertices, bipartite graphs, and Eulerian properties. Important theorems including Eu- ler's theorem and connectivity principles are applied to identify influential users and com- munication bottlenecks. The results show that the platforms exhibit struc- tures: WhatsApp networks are highly clustered and nearly connected. The study demonstrates that graph theory provides an ef- fective tool for understanding information flow, network stability, and influence patterns in social networks.

Key words: Graph Theory, Social Network Analysis, Structural Analysis, Degree Distribution, Connectivity, Directed and Undirected Graphs, Bridges and Cut-Vertices, Eulerian Graphs, Information Flow, Network Stability.

AMS classification: 05C78,05C90,

1 Introduction

Social networks play a crucial role in shaping human communication and interaction in modern society. With the rapid growth of online platforms such

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as Facebook, Twitter, and Instagram, social interactions have increasingly moved into digital spaces, generating large and complex networks of connected users[1]. These networks influence how information spreads, how opinions are formed, and how communities emerge[2]. Understanding the 1 structure of social networks has therefore become an important area of study across mathe- matics, computer science, and social sciences[3]. The complexity and large scale of online social networks require systematic and quantitative methods of analysis. Graph theory pro- videsapowerfulmathematicalframeworkformodelingandanalyzingsuchnetworks[4]. In this representation, users are considered as vertices and social relationships such as friend- ships or follower connections are represented as edges[5]. This abstraction allows complex social systems to be studied using well-defined mathematical tools and measures. Graph- theoretic concepts such as degree distribution, centrality measures, clustering coefficient, and shortest path length help reveal important structural properties of social networks[6]. These measures enable the identification of influential users, the detection of community structures, and the analysis of information flow within networks. This representation allows complex social systems to be analyzed using well-defined mathematical tools. Important concepts such as degree distribution, centrality measures, clustering coefficient, and short- estpathlengthhelprevealthestructuralpropertiesofnetworks. Real-worldsocialnetworks often exhibit special characteristics like small-world and scale-free properties. These fea- tures help in identifying influential users, detecting communities, and understanding the flow of information.The aim of this project is to perform a structural analysis of social net- works using graph theory by combining theoretical models with real-life oriented datasets. By analyzing data related to popular social media platforms, this study demonstrates the practical relevance of mathematical theory in understanding real-world social systems. The project highlights how graph theory serves as an effective tool for bridging abstract mathe- matical concepts with real social network behavior.

2 Definitions

Definition 2.1 Graph: A graph is a mathematical structure used to model pairwise relation ships between objects. Formally, a graph is defined as an ordered pair.
G = (V,E) where V = v 1 ,v 2 ,...,v n is a non-empty set of vertices (nodes), and E is a set of edges (links) connecting pairs of vertices. In social networks, vertices represent people or organizations, and edges represent relationships such

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as friendship, communication or collaboration[9].

Definition 2.2 Degree of a Vertex:

The degree of a vertex v, denoted by deg(v), is the number of edges incident to it[10]. In directed graphs, two types of degrees are defined: • In-degree: The number of incoming edges to a vertex. • Out-degree: The number of outgoing edges from a vertex.

Definition 2.3 Path: A path in a graph is a sequence of vertices$v_1, v_2, ..., v_k$such that each consecutive pair (v_i, v_{i+1}) is connected by an edge[**?**]. The length of a path is defined as the number of edges in the sequence. In social networks, a path represents a chain of users through which information can travel.

Definition 2.4 Connectivity: A graph is said to be connected if there exists at least one path between every pair of vertices in the graph[10]. If no such path exists for some pair of vertices, the graph is called disconnected. In social network analysis, connectivity indicates whether all users can reach one another directly or indirectly.

Definition 2.5 Bridge: An edge $e \in E$ in a graph G = (V,E) is called a bridge if its removal increases the number of connected components of the graph[10]. Formally, an edge e is a bridge if

$$c(G - e) > c(G)$$

where c(G) denotes the number of connected components of the graph. In social networks, a bridge represents a critical connection between two groups. Removing this connection can break communication between communities.

Definition 2.6 Cut-Vertex: A vertex $v \in V$ is called a cut-vertex (or articulation point) if its removal increases the number of connected components of the graph. Formally, a vertex v is a cut-vertex if

$$c(G - v) > c(G).$$

Definition 2.7 Euler Condition: A connected graph has an Euler circuit if and only if every vertex has an even degree. A connected graph has an Euler path if and only

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if exactly two vertices have odd degree, while all other vertices have even degree. If more than two vertices have odd degree, the graph has neither an Euler path nor an Euler circuit. In social networks, a cut-vertex represents an important user whose removal can disconnect groups or communities within the network.

3 EULER GRAPH APPLIED TO WHATSAPP MESSAGE PATH ANALYSIS

A WhatsApp group can be modeled as an undirected graph G = (V,E), where V represents the set of users and E represents message exchanges between users. A graph G is said to be an Euler graph (or Eulerian graph) if it is connected and every vertex has even degree:

$$G \text{ is Eulerian } \iff G \text{ is connected and } \deg(v) \equiv 0 \pmod 2, \quad \forall\, v \in V.$$

In a WhatsApp network, the degree of a vertex v is defined as

$$deg(v) = number of message interactions of user v.$$

Vertices(Nodes): The set of vertices is given by

$$V = \{a, b, c, d, e\}$$

Edges:

$$E = \{(a, b), (a, c), (c, d), (a, d), (c, b), (c, e), (d, e)\}$$

• Edge 1: Between a and b (horizontal bottom edge).
• Edge 2: Between a and c (vertical left edge).

• Edge 3: Between c and d (top horizontal edge). • Edge 4: Diagonal edges connecting a–d and c–b (cross connections). • Edge 5: Diagonal edges connecting c–e and d–e, indicating optional or weak links. **Graph:** Let G = (V,E) be a graph. where V is the set of users and E is the set of message interactions.

Euler Graph Condition:

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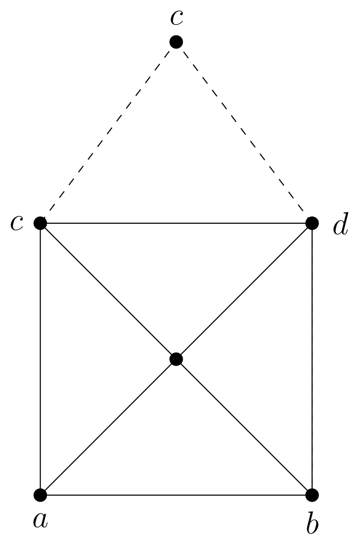

Figure 1: Network Structure Graph

A graph G is said to be Eulerian if:

$$G \text{ is Eulerian} \iff \begin{cases} G \text{ is connected,} \\ \\ \deg(v) \equiv 0 \pmod 2, \qquad \forall\, v \in V. \end{cases}$$

Degree of a Vertex: The degree of a vertex is defined as

$$\deg(v) = \text{number of message interactions of user} v.$$

If all users have even degrees, the network admits an Euler circuit, implying that message interactions can be traversed exactly once without interruption. This represents balanced participation among users in the group. The Euler graph condition helps identify whether Whatsapp message interac- tions are balanced and uninterrupted across the entire group.

4 BRIDGES AND CUTVERTICES IN SOCIAL GROUPS

Let a social network be represented by a graph G = (V,E) where V represents users (vertices) and E represents social interactions (edges). An edge e ∈E is called a bridge if its removal increases the number of connected components:

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$$c(G - e) > c(G),$$

where $c(G)$ denotes the number of connected components of the graph. Similarly, a vertex v ∈ V is called a cut-vertex if its removal disconnects the grap

$$c(G - v) > c(G)$$

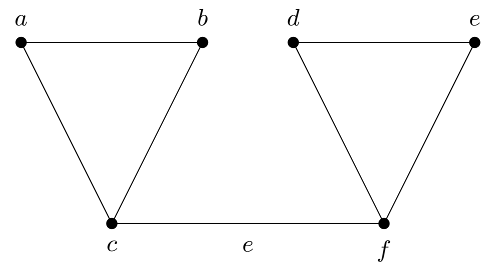

Figure 2: Bridge and Cut Vertex

The graph G = (V,E) is defined as follows

Vertices (V):

$$V = A, B, C, D, E, F$$

Edges (E):

- Group 1 (Clique): (A,B),(B,C),(A,C)
- Group 2 (Clique): (D,E),(E,F),(D,F)

Bridge: $e = (C, F)$ In graph theory, the given structure is a connected graph consisting of two 3-cliques. Vertices C and F act as cut-vertices, and the edge $e = (C, F)$ is a bridge. Removing e disconnects the graph into two components.

In social networks, a bridge represents a crucial communication link between commu- nities, and a cut-vertex represents a key user connecting multiple groups. Removing such edges or vertices fragments the network, indicating their structural and social significance.

Bridges and cut-vertices identify structurally important links and users whose removal causes social network fragmentation.

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5 STRUCTURAL PROPERTIES OF NETWORKS

Structural properties describe the overall shape and internal organization of a social net- work when it is represented as a graph $G = (V, E)$. These properties explain how users are connected, how tightly the network is bonded, and how resilient it is to failures. Degree structure is a primary property, where the degree of a vertex represents the number of direct connections a user has. The average degree of the network is given by

$$\bar{d} = \frac{2|E|}{|V|}$$

which indicates whether the network is dense or sparse. Connectivity is another key property that measures network robustness. Vertex connectivity $\kappa(G)$ and edge connectivity $\lambda(G)$ determine the minimum number of users or links whose removal can disconnect the network, satisfying the inequality

$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$

where $\delta(G)$ is the minimum degree of the graph. The presence of cut vertices and bridges reveals critical users and links that control network stability. Clustering and density de- scribe community formation and the closeness of the network to an ideal complete graph, while path length explains the efficiency of information flow.

5.1 Small-World Property

The small-world property is an important structural characteristic observed in many real- world social networks[14]. A network is said to exhibit the small-world property if it has a high clustering coefficient and a small average path length. This means that although individuals may belong to tightly connected groups or communities, any individual in the network can be reached from any other individual through only a few intermediate connec- tions. The concept of small-world networks was popularized by the idea of "six degrees of separation," which states that any two people in the world are connected by a chain of about six acquaintances. In social networks such as Facebook, WhatsApp, and Twitter, the small-world property enables rapid dissemination of information, rumors, and viral con- tent across the network. Mathematically, a network is considered a small-world network if its average

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path length is comparable to that of a random graph while its clustering co- efficient is significantly higher than that of a random graph. This property plays a crucial role in understanding communication efficiency, information spread, and social influence dynamics.

$$L \approx L_{\text{random}} \quad \text{and} \quad C \gg C_{\text{random}}$$

Where,

- L = Average path length of the given network
- C = Clustering coefficient of the given network
- L random = Average path length of an equivalent random graph
- C random = Clustering coefficient of an equivalent random graph

5.2 Scale Free Property

In graph theory, a network is said to exhibit the scale-free property when its degree dis- tribution follows a power-law behavior [13]. This means that most nodes have only a few connections, while a small number of nodes, called hubs, have a very large number of con- nections.Unlike regular or random graphs where node degrees are uniformly distributed, scale-free networks show a highly uneven degree distribution. Mathematically, the proba- bility that a node has degree k follows a power-law function given by

$$P(k) \sim k^{-\gamma}$$

Where,

- P(k) = Probability of a node having degree k
- k = Node degree
- γ = Power-law exponent

5.3 Network Density

Network density is a fundamental structural property that measures how strongly connected a network is [11]. It is defined as the ratio of the number of existing edges to the maximum possible number of edges in the network. A dense network indicates that most nodes are interconnected, whereas a sparse network indicates fewer connections among nodes. In social networks, high density implies strong social interaction and frequent communication among individuals. Density is mathematically expressed as

$$\text{Density} = \frac{2|E|}{|V|(|V|-1)}$$

where |V | is the number of vertices and |E| is the number of edges.

5.4 Average Path Length

Average path length represents the average number of steps required to connect any two nodes in the network using the shortest paths[12]. It reflects how quickly information can spread through the network. Social networks usually have small average path lengths, which leads to the concept of "six degrees of separation." A smaller average path length indicates efficient communication and faster diffusion of information.

$$L = \frac{1}{|V|(|V|-1)} \sum_{u \neq v} d(u, v)$$

where d(u,v) is the shortest path distance between nodes u and v.

5.5 Network Diameter

The diameter of a network is the longest shortest path between any pair of nodes[11]. It represents the maximum distance between two nodes in the network. Diameter is an im- portant structural property because it provides insight into the overall size and reachability of the network. A small diameter indicates that all nodes are relatively close to each other, while a large diameter indicates a more spread-out network. The diameter is the maximum shortest path distance in the network.

$$D = \max_{u,v \in V} d(u, v)$$

5.6 Connected Components

Connected components are subgraphs in which any two nodes are connected by at least one path[12]. In social networks, connected components represent separate social groups or communities that have no interaction with each other. Identifying connected compo- nents helps in understanding fragmented networks and isolated groups within large social systems. A connected component is a maximal subgraph in which every pair of nodes is connected by a path.

$$G = G_1 \cup G_2 \cup \cdots \cup G_N$$

where G i are the connected components.

5.7 Hubs and authorities

Hubs and Authorities are concepts introduced in the Hyperlink-Induced Topic Search(HITS) algorithm[17].

- Authority: A node that contains valuable or trusted information.
- Hub: A node that links to many authorities.

$$\text{Authority score:} \qquad a(v) = \sum_{u:(u,v)\in E} h(u)$$

$$\text{Hub score:} \qquad h(v) = \sum_{u:(v,u)\in E} a(u)$$

5.8 Community Detection

Community detection is the process of identifying groups of nodes that are densely con- nected internally but sparsely connected with other groups[16]. These communities repre- sent social groups, interest clusters, or functional units in a network.

$$Q = \frac{1}{2m} \sum_{i,j} \left(A_{ij} - \frac{K_i K_j}{2m} \right) \delta(C_i, C_j)$$

Where,

- A_{ij}= Entry of the adjacency matrix
- k_i, k_j = Degrees of vertices i and j

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- C_i, C_j = Community labels of vertices i and j

5.9 Link Analysis

Link analysis is a technique used to study the relationships (links) between entities (nodes) in a network[14]. It focuses on how nodes are connected and how these connections in- fluence importance, behavior, or information flow in the network. In social networks, link analysis helps identify influential users, hidden relationships, and structural patterns using graph theory. A social network is represented as a graph $G = (V, E)$ where, V = set of vertices (users) E = set of edges (links/relationships).

6 STRUCTURAL MODEL OF WHATSAPP

Let the platform be modeled as an undirected graph

$$G = (V, E),$$

where V represents the set of users and E represents communication links. Assume that the vertex set admits a partition

$$V = \bigcup_{i=1}^{k} V_i$$

such that

$$V_i \cap V_j = \emptyset \qquad (i \neq j).$$

Each V_i induces a subgraph

$$G_i = G[V_i],$$

which represents a communication group. If each G_i satisfies property (8.3)

$$D(G_i) = \frac{2|E_i|}{|V_i|\,(|V_i| - 1)} \approx 1,$$

then the group is densely connected. Further, if there exists a vertex

$$h_i \in V_i$$

.

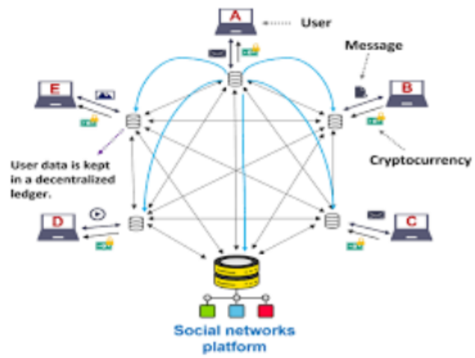

Figure 3: Decentralized Structure

such that

$$\deg(h_i) = \max_{v \in V_i} \deg(v),$$

then h_i acts as a local hub. Since edges between different V_i are minimal or absent, the global graph can be expressed as

$$G = \bigcup_{i=1}^{k} G_i$$

which implies that the network consists of multiple connected components.

Hence, the structure exhibits:

- High clustering within communities,
- Localized hub dominance,
- Weak global connectivity.

This Figure 3 shows the decentralized structure of a social network platform, where

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multiple users are connected through communication links. The diagram illustrates how users (vertices) interact with each other through messages, forming a network of connections (edges).

The network is divided into smaller groups or communities, where each group is densely connected internally. Therefore, the network is theoretically classified as a **Community-Based Decentralized Graph Structure**.

The structural analysis shows that the network exhibits:

High local density and clustering with weak inter-group connectivity, resulting in a globally decentralized but locally centralized architecture.

7 CONCLUSION

This study demonstrates that graph theory provides a powerful mathematical framework for analysing the structural characteristics of social networks. By representing users as vertices and their interactions as edges, various graph-theoretic concepts such as connectivity, Eulerian properties ,bridges,cut-vertices,degree distribution,hubs and authorities,community detection, and structural metrices were successfully applied to model and interpret social network behavier. The analysis reveals that social networks generally exhibit high lical clustering,decentralized community structures,influential hub nodes,and efficient communicationpaths. Furthermore it shows that graph theory effectively models and analyzes social networks by identifying critical users, communication links,and community structures. The proposed models deonstrate how graph-theoritic concepts can explain information flow , network stability, and communication patterns in platforms such as WhatsApp.Future work may extend this study by considering weighted, dynamic, and temporal networks using real-world datasets.

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