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Fixed Point Results for Generalized Rational Type $(\vartheta, \psi, \varphi)$-Weak Contractive Mappings in b-Metric Spaces with Application

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Abstract

This problem concentrates on existence and uniqueness of common fixed point for two pairs of self mappings gratify broad rational type $(\vartheta, \psi, \varphi)$ - weak contractive axiom in the framework of b-Metric Space. The pairs of mappings are weakly compatible and one pair is α - admissible w.r.t. the other. Our results broad and improve some well known results in the literature. As an application of our result, we shall discuss the existence of common solution of integral equations.

Key words: α**- admissible mapping, common fixed point, weakly compatible mappings, rational type** $(\vartheta, \psi, \varphi)$ **weak contractive condition, integral equation.**
MSC 2020: 47H10; 54H25.

1 Introduction

The metric fixed point(FP) theory progressed a lot after the significant result due to Banach[23] known as the Banach contraction mapping theorem. Bakhtin[24] initiated the study of a broad metric space (MS) named b -Metric Spaces (b-MS) and presented a version of the Banach contraction principle (BCP) in the context of b-MS. Subsequently several researchers studied FP theory for single valued and set valued mapping in b-MS see [13,12,11,23,26,5,27,20].
The concept of α-admissible and $(\alpha - \psi)$-contractive mappings introduced by Samet,Vetro and Vetro[5]. Then, La Rosa and Vetro[26] presented the notion of f- α-admissible mappings and obtained common fixed point(CFP) theorems for (α, ψ, φ)-contractions in broad MS. In 2018, Zoto and Vardhami[17] proved CFP results for broad α_{sp} contractive mappings on the setting of b-metric-like spaces using

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the concept of g - α_{s^p}-admissible mapping.
Numerous authors improved and modified the α-admissibility condition in the setting of various abstract spaces; see[21,3] and the references therein. Abbas, Chemac, and Razani[13] presented the conditions for the existence of a CFP for b-metric rational type contraction. Aydi, Bota, and Moradi[11] established CFP results for single and multivalued mappings satisfying a weak φ-contraction in b- MS.
Hao and Guan[28] introduced a class of broad weakly contractive mappings and obtained common fixed points(CFPs) by using different algorithms involving broad weakly contractive conditions in the framework of b- MS. In 2009, Doric[6] proved CFP theorems for broad $(\psi-\varphi)$-weakly contractive mappings, where $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ are functions such that ψ is increasing and continuous and φ is an increasing and lower semi-continuous such that $\psi(\xi) = \varphi(\xi) = 0$ iff $\xi = 0$. On the other hand, several authors obtained fixed points (FPs) and CFPs for a class of mappings satisfying certain contractive conditions and employed them to establish the existence of solutions of fractional and ordinary differential equations(ODE) and nonlinear integral equations(I.E.), see [12,18,8,20] and the references therein.

2 Preliminaries and Main Results

Definition 2.1 [24] Let $\mathfrak{M}_{\mathfrak{b}} \neq \phi$ set and $s \geq 1$ be a real number. A function $d : \mathfrak{M}_{\mathfrak{b}} \times \mathfrak{M}_{\mathfrak{b}} \to \mathbb{R}^+$ is said to b-MS

(i) $d(\zeta, \eta) = 0$ iff $\zeta = \eta$ for any $\zeta, \eta \in \mathfrak{M}_{\mathfrak{b}}$;
(ii) $d(\zeta, \eta) = d(\eta, \zeta)$;
(iii) $d(\zeta, \eta) \leq$ s $[d(\zeta, \kappa) + d(\kappa, \eta)]$, $\forall\ \zeta, \eta, \kappa \in \mathfrak{M}_{\mathfrak{b}}$.

The d is said to b-metric on $\mathfrak{M}_{\mathfrak{b}}$ and $(\mathfrak{M}_{\mathfrak{b}}, d, s)$ is said to b-MS with coefficients. b-MS is metric with $s = 1$.

Definition 2.2 [24] Let $(\mathfrak{M}_{\mathfrak{b}}, d, s)$ be a b-MS with $s \geq 1$ and $\{\zeta_n\}$ be a sequence in $\mathfrak{M}_{\mathfrak{b}}$. Then,

(i) $\{\zeta_n\}$ is said to b-convergent if $\exists\ \zeta \in \mathfrak{M}_{\mathfrak{b}}$ s.t. $d(\zeta_n, \zeta) \to 0$ as $n \to \infty$. In this case, we write $lim_{n\to\infty}\zeta_n = \zeta$.
(ii) $\{\zeta_n\}$ is said to b-Cauchy iff $lim_{n,m\to\infty}d(\zeta_n, \zeta_m) = 0$.

Definition 2.3 [24] A b-MS $(\mathfrak{M}_{\mathfrak{b}}, d, s)$, with $s \geq 1$, is known as complete if every

b-Cauchy sequence in $\mathfrak{M}_{\mathfrak{b}}$ is b-convergent.

Definition 2.4 [9] Two self- mappings $\mathfrak{T}$ and $\mathfrak{P}$ on $\mathfrak{M}_{\mathfrak{b}}$ are known as be weakly compatible if $\mathfrak{T}$ and $\mathfrak{P}$ commute at their coincidence points(CP), that is, $\mathfrak{T}\zeta = \mathfrak{P}\zeta$, $\forall\ \zeta \in \mathfrak{C}P(\mathfrak{T}, \mathfrak{P}) \neq \varphi$ [28] A function $\mathfrak{G} : \mathfrak{M}_{\mathfrak{b}} \to \mathbb{R}^{+}$ in a b-MS $(\mathfrak{M}_{\mathfrak{b}}, d, s)$ with $s \geq 1$ is said to lower semi-continuous if, for any sequence $\{\eta_n\}$ in $\mathfrak{M}_{\mathfrak{b}}$ which is b-convergent to $\eta \in \mathfrak{M}_{\mathfrak{b}}$, we have

$$\mathfrak{G}(\eta) \leq lim_{n\to\infty} \inf \mathfrak{G}(\eta_n).$$

Later Karapinar et al.[7], Shahi et al.[22], Abdeljawad[25] and Rao et al.[15] extended α-adimissible mappings connect with two and four mappings and established fixed and CFP theorems for mappings on several spaces.

Definition 2.5 Let $\mathfrak{X} \neq \phi$ set $\alpha : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^{+}$. We need the following definitions and notations in the rest of research see[25,7,15,5,22]

(i) A mapping of $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ is said to be α- admissible if $\alpha(\zeta, \eta) \geq 1 \implies \alpha(\mathfrak{T}\zeta, \mathfrak{T}\eta) \geq 1\ \forall\ \zeta, \eta \in \mathfrak{X}$.

(ii) A mapping of $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ is said to be triangular α- admissible if $\alpha(\zeta, \eta) \geq 1 \implies \alpha(\mathfrak{T}\zeta, \mathfrak{T}\eta) \geq 1\ \forall\ \zeta, \eta \in \mathfrak{X}$ and $\alpha(\zeta, \kappa) \geq 1$ and $\alpha(\kappa, \eta) \geq 1 \implies \alpha(\zeta, \eta) \geq 1$ $\forall\ \zeta, \eta, \kappa \in \mathfrak{X}$.

(iii) Let $\mathfrak{F}, \mathfrak{G} : \mathfrak{X} \to \mathfrak{X}$. Then $\mathfrak{F}$ is known as α- admissible w.r. to $\mathfrak{G}$ if $\alpha(\mathfrak{G}\zeta, \mathfrak{G}\eta) \geq 1 \implies \alpha(\mathfrak{F}\zeta, \mathfrak{F}\eta) \geq 1\ \forall\ \zeta, \eta \in \mathfrak{X}$.

(iv) Let $\mathfrak{F}, \mathfrak{G} : \mathfrak{X} \to \mathfrak{X}$. Then the pair $\mathfrak{F}, \mathfrak{G}$ is known as α- admissible if $\alpha(\zeta, \eta) \geq 1 \implies \alpha(\mathfrak{F}\zeta, \mathfrak{G}\eta) \geq 1$ and $\alpha(\mathfrak{G}\zeta, \mathfrak{F}\eta) \geq 1\ \forall\ \zeta, \eta \in \mathfrak{X}$.

(v) Let $\mathfrak{F}, \mathfrak{G}, \mathfrak{P}, \mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$. Then the pair $(\mathfrak{F}, \mathfrak{G})$ is known as α- admissible w.r.t. the pair $(\mathfrak{P}, \mathfrak{T})$ if $\alpha(\mathfrak{P}\zeta, \mathfrak{T}\eta) \geq 1 \implies \alpha(\mathfrak{F}\zeta, \mathfrak{G}\eta) \geq 1$ and $\alpha(\mathfrak{T}\zeta, \mathfrak{P}\eta) \geq 1 \implies \alpha(\mathfrak{G}\zeta, \mathfrak{F}\eta) \geq 1\ \forall\ \zeta, \eta \in \mathfrak{X}$.

Lemma 2.6 [14] Let $(\mathfrak{M}_{\mathfrak{b}}, d, s)$ be a b-MS with $s \geq 1$ and let $\{\eta_n\}$ be a sequence in $\mathfrak{M}_{\mathfrak{b}}$ s.t.

$$lim_{n\to\infty} d(\eta_n, \eta_{n+1}) = 0$$

If the sequence $\{\eta_n\}$ is not b-cauchy, then $\exists\ \epsilon > 0$ for which one can find two subsequences $\{\eta_{n_j}\}$ and $\{\eta_{m_j}\}$ of $\{\eta_n\}$ s.t. $n_j > m_j > j$ and for the following four sequences

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$\Big\{d(\eta_{m_j},\eta_{n_j})\Big\}$, $\Big\{d(\eta_{m_j-1},\eta_{n_j-1})\Big\}$, $\Big\{d(\eta_{m_j-1},\eta_{n_j})\Big\}$, $\Big\{d(\eta_{m_j},\eta_{n_j-1})\Big\}$ it holds that:

$$\epsilon \leq lim_{n\to\infty} inf\, d(\eta_{m_j},\eta_{n_j}) \leq lim_{n\to\infty}\sup d(\eta_{m_j},\eta_{n_j}) \leq s\epsilon;$$

$$\frac{\epsilon}{s} \leq lim_{n\to\infty} inf\, d(\eta_{m_j},\eta_{n_j-1}) \leq lim_{n\to\infty}\sup d(\eta_{m_j},\eta_{n_j-1}) \leq \epsilon;$$

$$\frac{\epsilon}{s} \leq lim_{n\to\infty}\inf d(\eta_{m_j-1},\eta_{n_j}) \leq lim_{n\to\infty}\sup d(\eta_{m_j-1},\eta_{n_j}) \leq s^2\epsilon;$$

$$\frac{\epsilon}{s^2} \leq lim_{n\to\infty}\inf d(\eta_{m_j-1},\eta_{n_j-1}) \leq lim_{n\to\infty}\sup d(\eta_{m_j-1},\eta_{n_j-1}) \leq s\epsilon.$$

Lemma 2.7 [1] Let $(\mathfrak{M}_\mathfrak{b}, d, s)$ be a b-MS with $s \geq 1$ and let $\{\zeta_n\}$ and $\{\eta_n\}$ be b-convergent to points $\zeta, \eta \in \mathfrak{M}_\mathfrak{b}$, respectively. Then, we have,

$$\frac{1}{s^2}d(\zeta,\eta) \leq lim_{n\to\infty}\inf d(\zeta_n,\eta_n) \leq lim_{n\to\infty}\sup d(\zeta_n,\eta_n) \leq s^2 d(\zeta,\eta).$$

In particular, if $\zeta = \eta$. then $lim_{n\to\infty} d(\zeta_n,\eta_n) = 0$. Moreover, for every $\omega \in \mathfrak{M}_\mathfrak{b}$, we have,

$$\frac{1}{s^2}d(\zeta,\omega) \leq lim_{n\to\infty}\inf d(\zeta_n,\omega) \leq lim_{n\to\infty}\sup d(\zeta_n,\omega) \leq s d(\zeta,\omega).$$

We formulated the contractive axiom in our main results based on the following set of functions:

$$\Theta = \left\{\vartheta : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is an increasing and continuous function s.t } \vartheta(\xi) = 0 \text{ iff } \xi = 0\right\};$$

$$\Psi = \Big\{\psi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is an increasing and lower semi- continuous function s.t. } \psi(\xi) = 0 \text{ iff } \xi = 0\Big\};$$

$$\Phi = \left\{\varphi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is a continuous mapping s.t. } \varphi(\xi) = 0 \text{ iff } \xi = 0\right\};$$

In 2025, Kalo et al.[2] introduced broad rational type $(\vartheta, \varphi, \psi)$ weak contractive mappings and studied CFP for two mappings in the framework of b-MS.

Theorem 2.8 [2] Let $(\mathfrak{M}_\mathfrak{b}, d, s)$ be a b-complete metric space (b-CMS) with $s \geq 1$ and let α:$\mathfrak{M}_\mathfrak{b} \times \mathfrak{M}_\mathfrak{b} \to \mathbb{R}^+$ be a function. Suppose $\mathfrak{T}, \mathfrak{P} : \mathfrak{M}_\mathfrak{b} \to \mathfrak{M}_\mathfrak{b}$ are two self-mappings s.t. $\mathfrak{T}$ is $(\mathfrak{P}$-$\alpha_s)$ - admissible and gratifying the following axioms:

(i) $\vartheta(\alpha(\mathfrak{P}\zeta,\mathfrak{T}\eta)d(\mathfrak{F}\zeta,\mathfrak{G}\eta)) \leq \vartheta(\mathfrak{M}_{\mathfrak{b}_\mathfrak{i}}(\zeta,\eta)) - \psi(\mathfrak{M}_{\mathfrak{b}_\mathfrak{i}}(\zeta,\eta)) + \gamma\varphi(\mathfrak{N}(\zeta,\eta))\,\forall\,\zeta,\eta \in \mathfrak{M}_\mathfrak{b}$ with $\alpha(\mathfrak{P}\zeta,\mathfrak{T}\eta) \geq s$, where i=1,2

(ii) the pair $(\mathfrak{T},\mathfrak{P})$ is a broad rational type (ϑ,ψ,φ) weak contraction mapping with

$$\mathfrak{M}_{\mathfrak{b}_1}(\zeta,\eta) = max\Big\{d(\mathfrak{P}\zeta,\mathfrak{P}\eta), d(\mathfrak{T}\zeta,\mathfrak{P}\zeta), d(\mathfrak{T}\eta,\mathfrak{P}\eta), \frac{1}{2s}[d(\mathfrak{T}\zeta,\mathfrak{P}\eta)+d(\mathfrak{T}\eta,\mathfrak{P}\zeta)], \frac{d(\mathfrak{T}\eta,\mathfrak{P}\eta)[1+d(\mathfrak{T}\zeta,\mathfrak{P}\zeta)]}{1+d(\mathfrak{P}\zeta,\mathfrak{P}\eta)}\Big\}$$

$$\mathfrak{M}_{\mathfrak{b}_2}(\zeta,\eta) = max\Big\{d(\mathfrak{P}\zeta,\mathfrak{P}\eta), d(\mathfrak{T}\zeta,\mathfrak{P}\zeta), d(\mathfrak{T}\eta,\mathfrak{P}\eta), \frac{d(\mathfrak{T}\zeta,\mathfrak{P}\zeta)d(\mathfrak{T}\eta,\mathfrak{P}\zeta)+d(\mathfrak{T}\eta,\mathfrak{P}\eta)d(\mathfrak{T}\zeta,\mathfrak{P}\eta)}{1+d(\mathfrak{T}\eta,\mathfrak{P}\zeta)+d(\mathfrak{T}\zeta,\mathfrak{P}\eta)}, \frac{d(\mathfrak{T}\zeta,\mathfrak{P}\zeta)d(\mathfrak{T}\eta,\mathfrak{P}\zeta)+d(\mathfrak{T}\eta,\mathfrak{P}\eta)d(\mathfrak{T}\zeta,\mathfrak{P}\eta)}{1+s(d(\mathfrak{T}\zeta,\mathfrak{P}\zeta)+d(\mathfrak{T}\eta,\mathfrak{P}\eta)}\Big\},$$

$$\mathfrak{N}(\zeta,\eta) = min\left\{d(\mathfrak{P}\zeta,\mathfrak{F}\zeta), d(\mathfrak{T}\eta,\mathfrak{G}\eta), d(\mathfrak{P}\zeta,\mathfrak{G}\eta), d(\mathfrak{T}\eta,\mathfrak{F}\zeta)\right\}.$$

(iii) $\mathfrak{T}(\mathfrak{M}_\mathfrak{b}) \subseteq \mathfrak{P}(\mathfrak{M}_\mathfrak{b})$ and $\mathfrak{P}(\mathfrak{M}_\mathfrak{b})$ is a closed subspace of $\mathfrak{M}_\mathfrak{b}$;

(iv) $\exists\ \zeta_0 \in \mathfrak{M}_\mathfrak{b}$ such that $\alpha(\mathfrak{P}\zeta_0,\mathfrak{T}\zeta_0) \geq s$;

(v) for any non-decreasing sequence $\{\mathfrak{P}\zeta_n\}$ in $\mathfrak{M}_\mathfrak{b}$ with $\mathfrak{P}\zeta_n \to \mathfrak{P}\zeta$ in $\mathfrak{P}(\mathfrak{M}_\mathfrak{b})$ as $n \to \infty$ and $\alpha(\mathfrak{P}\zeta_n,\mathfrak{P}\zeta_{n+1}) \geq s\ \forall\ n \in \mathbb{N}_0$, we have $\alpha(\mathfrak{P}\zeta_n,\mathfrak{P}\zeta) \geq s\ \forall\ n \in \mathbb{N}_0$;

(vi) $\alpha(\mathfrak{P}\varpi,\mathfrak{P}\kappa) \geq s\ \forall\ \varpi,\kappa \in \mathfrak{C}P(\mathfrak{T},\mathfrak{P})$.

Then $\mathfrak{T}$ and $\mathfrak{P}$ admit unique coincidence point(CP)in $\mathfrak{M}_\mathfrak{b}$. Moreever, $\mathfrak{T}$ and $\mathfrak{P}$ have unique CFP in $\mathfrak{M}_\mathfrak{b}$ provided that $\mathfrak{T}$ and $\mathfrak{P}$ are weakly compatible.
Inspired by the results of Kalo et al.[2], We define broad rational type (ϑ,ψ,φ) weak contractive condition for two pairs of self mappings.

Definition 2.9 Let $(\mathfrak{M}_\mathfrak{b},d,s)$ be a b-MS with $s \geq 1$ and let α:$\mathfrak{M}_\mathfrak{b} \times \mathfrak{M}_\mathfrak{b} \to \mathbb{R}^+$ be a function. Let $\mathfrak{T},\mathfrak{P},\mathfrak{F},\mathfrak{G} : \mathfrak{M}_\mathfrak{b} \to \mathfrak{M}_\mathfrak{b}$ be mappings s.t. $(\mathfrak{F},\mathfrak{G})$ is α- admissible w.r.t. $(\mathfrak{P},\mathfrak{T})$.
The pairs $(\mathfrak{T},\mathfrak{P})$ and $(\mathfrak{F},\mathfrak{G})$ are called broad rational type (ϑ,ψ,φ) weak contraction

mapping if $\exists \ \ \vartheta \in \vartheta$, $\psi \in \Psi$, $\varphi \in \Phi$ and $\gamma \geq 0$ s.t.

$$\vartheta(\alpha(\mathfrak{P}\zeta, \mathfrak{T}\eta)d(\mathfrak{F}\zeta, \mathfrak{G}\eta)) \leq \vartheta(\mathfrak{M}_{\mathfrak{b}_i}(\zeta, \eta)) - \psi(\mathfrak{M}_{\mathfrak{b}_i}(\zeta, \eta)) + \gamma\varphi(\mathfrak{N}(\zeta, \eta)) \text{ for all } \zeta, \eta \in \mathfrak{M}_{\mathfrak{b}} \tag{1}$$

$$\text{with } \alpha(\mathfrak{P}\zeta, \mathfrak{T}\eta) \geq s, \text{ where } i = 1, 2$$

and

$$\mathfrak{M}_{\mathfrak{b}_1}(\zeta, \eta) = max\Big\{d(\mathfrak{P}\zeta, \mathfrak{T}\eta), d(\mathfrak{P}\zeta, \mathfrak{F}\zeta), d(\mathfrak{T}\eta, \mathfrak{G}\eta), \frac{1}{2s}[d(\mathfrak{P}\zeta, \mathfrak{G}\eta) + d(\mathfrak{T}\eta, \mathfrak{F}\zeta)], \frac{d(\mathfrak{T}\eta, \mathfrak{G}\eta)[1 + d(\mathfrak{P}\zeta, \mathfrak{F}\zeta)]}{1 + d(\mathfrak{P}\zeta, \mathfrak{T}\eta)}\Big\}$$

$$\mathfrak{M}_{\mathfrak{b}_2}(\zeta, \eta) = max\Big\{d(\mathfrak{P}\zeta, \mathfrak{T}\eta), d(\mathfrak{P}\zeta, \mathfrak{F}\zeta), d(\mathfrak{T}\eta, \mathfrak{G}\eta), \frac{d(\mathfrak{P}\zeta, \mathfrak{F}\zeta)d(\mathfrak{T}\eta, \mathfrak{F}\zeta) + d(\mathfrak{T}\eta, \mathfrak{G}\eta)d(\mathfrak{P}\zeta, \mathfrak{G}\eta)}{1 + d(\mathfrak{T}\eta, \mathfrak{F}\zeta) + d(\mathfrak{P}\zeta, \mathfrak{G}\eta)}, \frac{d(\mathfrak{P}\zeta, \mathfrak{F}\zeta)d(\mathfrak{T}\eta, \mathfrak{F}\zeta) + d(\mathfrak{T}\eta, \mathfrak{G}\eta)d(\mathfrak{P}\zeta, \mathfrak{G}\eta)}{1 + s(d(\mathfrak{P}\zeta, \mathfrak{F}\zeta) + d(\mathfrak{T}\eta, \mathfrak{G}\eta))}\Big\},$$

$$\mathfrak{N}(\zeta, \eta) = min\left\{d(\mathfrak{P}\zeta, \mathfrak{F}\zeta), d(\mathfrak{T}\eta, \mathfrak{G}\eta), d(\mathfrak{P}\zeta, \mathfrak{G}\eta), d(\mathfrak{T}\eta, \mathfrak{F}\zeta)\right\}.$$

In this paper we establish CFP theorem for two pairs of weakly compatible mappings satisfying broad rational type $(\vartheta, \psi, \varphi)$ - weak contractive condition.

Theorem 2.10 Let $(\mathfrak{M}_{\mathfrak{b}}, d, s)$ be a b-CMS with $s \geq 1$ and let $\alpha{:}\mathfrak{M}_{\mathfrak{b}} \times \mathfrak{M}_{\mathfrak{b}} \to \mathbb{R}^+$ be a function. Suppose $\mathfrak{T}, \mathfrak{P}, \mathfrak{F}, \mathfrak{G} : \mathfrak{M}_{\mathfrak{b}} \to \mathfrak{M}_{\mathfrak{b}}$ are mappings s.t. $(\mathfrak{F}, \mathfrak{G})$ is α- admissible w.r.t. $(\mathfrak{P}, \mathfrak{T})$ and satisfying the following axioms:

(i) the pairs $(\mathfrak{T}, \mathfrak{P})$ and $(\mathfrak{F}, \mathfrak{G})$ are broad rational type $(\vartheta, \psi, \varphi)$ weak contraction mapping with $\mathfrak{M}_{\mathfrak{b}_i} = \mathfrak{M}_{\mathfrak{b}_1}$;
(ii) $\mathfrak{F}(\mathfrak{M}_{\mathfrak{b}}) \subseteq \mathfrak{T}(\mathfrak{M}_{\mathfrak{b}})$, $\mathfrak{G}(\mathfrak{M}_{\mathfrak{b}}) \subseteq \mathfrak{P}(\mathfrak{M}_{\mathfrak{b}})$ and $\mathfrak{T}(\mathfrak{M}_{\mathfrak{b}})$ or $\mathfrak{P}(\mathfrak{M}_{\mathfrak{b}})$ is a closed subspace of $\mathfrak{M}_{\mathfrak{b}}$;
(iii) $\exists\ \zeta_1 \in \mathfrak{M}_{\mathfrak{b}}$ s.t. $\alpha(\mathfrak{P}\zeta_1, \mathfrak{F}\zeta_1) \geq s$ and $\alpha(\mathfrak{F}\zeta_1, \mathfrak{P}\zeta_1) \geq s$;
(iv) for any non-decreasing sequence $\{\mathfrak{P}\zeta_n\}$ in $\mathfrak{M}_{\mathfrak{b}}$ with $\mathfrak{P}\zeta_n \to \mathfrak{P}\zeta$ in $\mathfrak{P}(\mathfrak{M}_{\mathfrak{b}})$ as $n \to \infty$ and $\alpha(\mathfrak{P}\zeta_n, \mathfrak{P}\zeta_{n+1}) \geq s\ \forall\ n \in \mathbb{N}_0$, we have $\alpha(\mathfrak{P}\zeta_n, \mathfrak{P}\zeta) \geq s\ \forall\ n \in \mathbb{N}_0$;
(v) $\alpha(\mathfrak{P}\kappa, \mathfrak{T}\varpi) \geq s\ \forall\ \varpi, \kappa \in \mathfrak{C}P(\mathfrak{P}, \mathfrak{T})$.

Then $\mathfrak{T}, \mathfrak{P}, \mathfrak{F}, \mathfrak{G}$ admit unique CP in $\mathfrak{M}_{\mathfrak{b}}$. Moreever, $(\mathfrak{T}, \mathfrak{P})$ and $(\mathfrak{F}, \mathfrak{G})$ have unique CFP in $\mathfrak{M}_{\mathfrak{b}}$ provided that $(\mathfrak{F}, \mathfrak{P})$ and $(\mathfrak{G}, \mathfrak{T})$ are weakly compatible.
Proof: From (iii) $\exists\ \zeta_1 \in \mathfrak{M}_{\mathfrak{b}}$ s.t. $\alpha(\mathfrak{P}\zeta_1, \mathfrak{F}\zeta_1) \geq s$ and $\alpha(\mathfrak{F}\zeta_1, \mathfrak{P}\zeta_1) \geq s$. Since

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$\mathfrak{F}(\mathfrak{M}_{\mathfrak{b}}) \subseteq \mathfrak{T}(\mathfrak{M}_{\mathfrak{b}})$, $\mathfrak{G}(\mathfrak{M}_{\mathfrak{b}}) \subseteq \mathfrak{P}(\mathfrak{M}_{\mathfrak{b}})$, $\exists$ sequences $\{\zeta_n\}$ and $\{\eta_n\}$ in $\mathfrak{M}_{\mathfrak{b}}$ s.t.

$$\eta_{2n+1} = \mathfrak{F}\zeta_{2n+1} = \mathfrak{T}\zeta_{2n+2} \text{ and } \eta_{2n+2} = \mathfrak{G}\zeta_{2n+2} = \mathfrak{P}\zeta_{2n+3}, n \in \mathbb{N}_0 \tag{2}$$

Now we have
$\alpha(\mathfrak{P}\zeta_1, \mathfrak{F}\zeta_1) \geq s \implies \alpha(\mathfrak{P}\zeta_1, \mathfrak{T}\zeta_2) \geq s,$

$$\implies \alpha(\mathfrak{F}\zeta_1, \mathfrak{G}\zeta_2) \geq s, i.e. \alpha(\eta_1, \eta_2) \geq s$$

$$\implies \alpha(\mathfrak{T}\zeta_2, \mathfrak{P}\zeta_3) \geq s,$$

$$\implies \alpha(\mathfrak{G}\zeta_2, \mathfrak{F}\zeta_3) \geq s, i.e. \alpha(\eta_2, \eta_3) \geq s$$

$$\implies \alpha(\mathfrak{P}\zeta_3, \mathfrak{T}\zeta_4) \geq s$$

$$\implies \alpha(\mathfrak{F}\zeta_3, \mathfrak{G}\zeta_4) \geq s, i.e. \alpha(\eta_3, \eta_4) \geq s$$

Continuing in this way,we have

$$\alpha(\eta_n, \eta_{n+1}) \geq s \, \forall \, n \in \mathbb{N} \tag{3}$$

Similarly by using $\alpha(\mathfrak{F}\zeta_1, \mathfrak{P}\zeta_1) \geq s$ we can claim that

$$\alpha(\eta_{n+1}, \eta_n) \geq s \, \forall \, n \in \mathbb{N}$$

From (2), we have if $\eta_{2n+1} = \eta_{2n+2}$ for some $n \in \mathbb{N}$ then $\mathfrak{T}\zeta_{2n+2} = \eta_{2n+1} = \eta_{2n+2} = \mathfrak{F}\zeta_{2n+2}$ and from (2)
If $\eta_{2n+2} = \eta_{2n+3}$ for some $n \in \mathbb{N}$ then $\mathfrak{P}\zeta_{2n+3} = \eta_{2n+2} = \eta_{2n+3} = \mathfrak{G}\zeta_{2n+3}$ $\forall$ $n \in \mathbb{N}_0$, that $(\mathfrak{P}, \mathfrak{G})$ and $(\mathfrak{T}, \mathfrak{F})$ have a CP at ζ_{2n+2} and ζ_{2n+3} and we have finished the proof so we assume that $\eta_n \neq \eta_{n+1}$ $\forall$ $n \in \mathbb{N}_0$.
from (3), $\alpha(\mathfrak{P}\zeta_{2n+1}, \mathfrak{T}\zeta_{2n+2}) = \alpha(\eta_{2n}, \eta_{2n+1}) \geq s$
from (1) we have
Case I. If n is odd i.e., n = 2n+1, n = 0,1,2,.... then we get

$$\vartheta(sd(\eta_{2n+1}, \eta_{2n+2}) = \vartheta(sd(\mathfrak{F}\zeta_{2n+1}, \mathfrak{G}\zeta_{2n+2})$$

$$\leq \vartheta(\alpha(\mathfrak{P}\zeta_{2n+1}, \mathfrak{T}\zeta_{2n+2})d(\mathfrak{F}\zeta_{2n+1}, \mathfrak{G}\zeta_{2n+2})$$

$$\leq \vartheta(\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{2n+1}, \zeta_{2n+2})) - \psi(\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{2n+1}, \zeta_{2n+2})) + \gamma\varphi(\mathfrak{N}(\zeta_{2n+1}, \zeta_{2n+2})) \tag{4}$$

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where
$\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{2n+1}, \zeta_{2n+2}) = \max\Big\{d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{T}\zeta_{2n+2}), d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{F}\zeta_{2n+1}), d(\mathfrak{T}\zeta_{2n+2}, \mathfrak{G}\zeta_{2n+2}),$
$\frac{1}{2s}[d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{G}\zeta_{2n+2}) + d(\mathfrak{T}\zeta_{2n+2}, \mathfrak{F}\zeta_{2n+1})], \frac{d(\mathfrak{T}\zeta_{2n+2}, \mathfrak{G}\zeta_{2n+2})[1+d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{F}\zeta_{2n+1})]}{1+d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{T}\zeta_{2n+2})}\Big\}$

$$\mathfrak{N}(\zeta_{2n+1}, \zeta_{2n+2}) = min\Big\{d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{F}\zeta_{2n+1}), d(\mathfrak{T}\zeta_{2n+2}, \mathfrak{G}\zeta_{2n+2}), d(\mathfrak{P}x_{2n_+1},$$

$$\mathfrak{G}\zeta_{2n+2}), d(\mathfrak{T}\zeta_{2n+2}, \mathfrak{F}\zeta_{2n+1})\Big\}$$

$$\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{2n+1}, \zeta_{2n+2}) = \max\Big\{d(\eta_{2n}, \eta_{2n+1}), d(\eta_{2n}, \eta_{2n+1}), d(\eta_{2n+1}, \eta_{2n+2}),$$

$$\frac{1}{2s}[d(\eta_{2n}, \eta_{2n+2}) + d(\eta_{2n+1}, \eta_{2n+1})], \frac{d(\eta_{2n+1}, \eta_{2n+2})[1 + d(\eta_{2n}, \eta_{2n+1})]}{1 + d(\eta_{2n}, \eta_{2n+1})}\Big\}$$

$$\leq max\left\{d(\eta_{2n}, \eta_{2n+1}), d(\eta_{2n+1}, \eta_{2n+2}), \frac{1}{2}[d(\eta_{2n}, \eta_{2n+1}) + d(\eta_{2n+1}, \eta_{2n+2})]\right\} \tag{5}$$

$$\mathfrak{N}(\zeta_{2n+1}, \zeta_{2n+2}) = min\{d(\eta_{2n}, \eta_{2n+1}), d(\eta_{2n+1}, \eta_{2n+2}), d(\eta_{2n}, \eta_{2n+2}), d(\eta_{2n+1}, \eta_{2n+1})\} = 0 \tag{6}$$

If $d(\eta_{2n+1}, \eta_{2n+2}) > d(\eta_{2n}, \eta_{2n+1})$ for some $n \in \mathbb{N}$, then by equations (4)-(6), we have

$$\begin{aligned}
\vartheta(d(\eta_{2n+1}, \eta_{2n+2}) &\leq \vartheta(sd(\eta_{2n+1}, \eta_{2n+2}) \\
&\leq \vartheta(\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{2n+1}, \zeta_{2n+2})) - \psi(\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{2n+1}, \zeta_{2n+2})) + \gamma\varphi(0) \\
&\leq \vartheta(\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{2n+1}, \zeta_{2n+2}) \\
&< \vartheta(d(\eta_{2n+1}, \eta_{2n+2})
\end{aligned}$$

which is contradictory. Thus from equation (5) we have

$$d(\eta_{2n+1}, \eta_{2n+2}) \leq \mathfrak{M}_{\mathfrak{b}_1}(\zeta_{2n+1}, \zeta_{2n+2}) \leq d(\eta_{2n}, \eta_{2n+1}). \tag{7}$$

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Case II. If n is even then similar case I. Hence $d(\eta_n, \eta_{n+1})$ is decreasing sequence.
Let $lim_{n\to\infty} d(\eta_n, \eta_{n+1}) = \lambda$, for some $\lambda \geq 0$.
From equation (7) it follows that

$$lim_{n\to\infty} d(\eta_{2n+1}, \eta_{2n+2}) = lim_{n\to\infty} \mathfrak{M}_{\mathfrak{b}_1}(\zeta_{2n+1}, \zeta_{2n+2}) = \lambda. \tag{8}$$

We claim $\lambda = 0$. Suppose that $\lambda > 0$. Approaching as $n \to \infty$ on both sides of the Equation (4) together with Equation (6) and (8), we have

$$= lim_{n\to\infty} \vartheta(sd(\eta_{2n+1}, \eta_{2n+2})$$

$$\vartheta(\lambda) \leq \vartheta(s\lambda) \leq lim_{n\to\infty} \Big\{ \vartheta(\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{2n+1}, \zeta_{2n+2})) - \psi(\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{2n+1}, \zeta_{2n+2}))$$

$+$

$$\gamma\varphi(\mathfrak{N}(\zeta_{2n+1}, \zeta_{2n+2})\Big\}$$

$$\leq \vartheta(\lambda) - \psi(\lambda) + \gamma\varphi(0)$$

$$\leq \vartheta(\lambda)$$

which is contradictory thus, $\lambda = 0$ and hence

$$lim_{n\to\infty} d(\eta_n, \eta_{n+1}) = 0 \tag{9}$$

Next, we claim that the sequence $\{\eta_n\}$ is b-cauchy in $\mathfrak{M}_{\mathfrak{b}}$. Assume, that $\{\eta_n\}$ is not b-cauchy . By lemma 2.1, $\exists\ \epsilon > 0$ and two subsequences $\{\eta_{n_j}\}$ and $\{\eta_{m_j}\}$ of $\{\eta_n\}$ $n_j > m_j \geq j$ for all $j \in \mathbb{N}$ and $d(\eta_{m_j}, \eta_{n_j}) \geq \epsilon$, $d(\eta_{m_j}, \eta_{n_j-1}) < \epsilon$
Since $n_j > m_j\ \forall\ j \in \mathbb{N}$ using equation (3) we have $\alpha(\mathfrak{P}\zeta_{m_j}, \mathfrak{T}\zeta_{n_j}) \geq s$. Now, putting $\zeta = \zeta_{m_j}$ and $\eta = \zeta_{n_j}$ in equation (1), we derive

$$\vartheta(sd(\eta_{m_j}, \eta_{n_j})) = \vartheta(sd(\mathfrak{F}\zeta_{m_j}, \mathfrak{G}\zeta_{n_j}))$$

$$\leq \vartheta(\alpha(\mathfrak{P}\zeta_{m_j}, \mathfrak{T}\zeta_{n_j}) d(\mathfrak{F}\zeta_{m_j}, \mathfrak{G}\zeta_{n_j}))$$

$$\leq \vartheta(d(\mathfrak{P}\zeta_{m_j}, \mathfrak{T}\zeta_{n_j})) - \psi(d(\mathfrak{P}\zeta_{m_j}, \mathfrak{T}\zeta_{n_j}))$$

$$\leq \vartheta(\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{m_j}, \zeta_{n_j})) - \psi(\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{m_j}, \zeta_{n_j})) + \gamma\varphi(\mathfrak{N}(\zeta_{m_j}, \zeta_{n_j})) \tag{10}$$

where

$$\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{m_j}, \zeta_{n_j}) = max\Big\{d(\mathfrak{P}\zeta_{m_j}, \mathfrak{T}\zeta_{n_j}), d(\mathfrak{P}\zeta_{m_j}, \mathfrak{F}\zeta_{m_j}), d(\mathfrak{T}\zeta_{n_j}, \mathfrak{G}\zeta_{n_j}),$$

$$\frac{1}{2s}[d(\mathfrak{P}\zeta_{m_j}, \mathfrak{G}\zeta_{n_j}) + d(\mathfrak{T}\zeta_{n_j}, \mathfrak{F}\zeta_{m_j})], \frac{d(\mathfrak{T}\zeta_{n_j}, \mathfrak{G}\zeta_{n_J}[1 + d(\mathfrak{P}\zeta_{m_j}, \mathfrak{F}\zeta_{m_j})]}{1 + d(\mathfrak{P}\zeta_{m_j}, \mathfrak{T}\zeta_{n_j})}\Big\}$$

$$= max\Big\{d(\eta_{m_j-1}, \eta_{n_j-1}), d(\eta_{m_j}, \eta_{m_j-1}), d(\eta_{n_j}, \eta_{n_j-1}), \frac{1}{2s}[d(\eta_{m_j}, \eta_{n_j-1}) + d(\eta_{n_j-1}, \eta_{m_j})], \tag{11}$$

$$\frac{d(\eta_{n_j}, \eta_{n_j-1})[1 + d(\eta_{m_j}, \eta_{m_j-1})]}{1 + d(\eta_{m_j-1} + \eta_{n_j-1})}\}\Big\}$$

$$\mathfrak{N}(\zeta_{m_j}, \zeta_{n_j}) = min\Big\{d(\mathfrak{P}\zeta_{m_j}, \mathfrak{F}\zeta_{n_j}), d(\mathfrak{T}\zeta_{n_j}, \mathfrak{G}\zeta_{n_j}), d(\mathfrak{P}\zeta_{m_j}, \mathfrak{G}\zeta_{n_j}), \tag{12}$$

$$d(\mathfrak{T}\zeta_{m_j}, \mathfrak{G}\zeta_{n_j}), d(\mathfrak{T}\zeta_{n_j}, \mathfrak{F}\zeta_{m_j})\Big\}$$

$$= min\Big\{d(\eta_{m_j}, \eta_{m_j-1}), d(\eta_{n_j}, \eta_{n_j-1}), d(\eta_{m_j}, \eta_{n_j-1}), d(\eta_{n_j}, \eta_{m_j-1})\Big\}$$

Now, approaching as $j \to \infty$ in equation (11) together with lemma 2.1 and using equation (9), we have

$$lim_{j\to\infty} \sup \mathfrak{M}_{\mathfrak{b}_1}(\zeta_{m_j}, x_{n_j}) \leq max\,\{s\epsilon, 0, 0, 0, 0\} = s\epsilon, \tag{13}$$

and

$$s\epsilon \geq lim_{j\to\infty} \inf \mathfrak{M}_{\mathfrak{b}_1}(\zeta_{m_j}, \zeta_{n_j}) \geq \frac{\epsilon}{s^2}. \tag{14}$$

Also,approaching as $j \to \infty$ in equation (12) together with lemma 2.1 and using equation (9), we have

$$0 \leq lim_{j\to\infty} \sup \mathfrak{N}(\zeta_{m_j}, \zeta_{n_j}) \leq min\,\Big\{0, 0, \epsilon, s^2\epsilon\Big\} = 0. \tag{15}$$

Approaching sup in equation (10) together with lemma 2.1 and using equation (13) and (15), we have

$$\vartheta(s\epsilon) \leq lim_{j\to\infty} \sup \vartheta(sd(\eta_{m_j}, \eta_{n_j}))$$

$$\leq lim_{j\to\infty} \sup \big\{\vartheta(\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{m_j}, \zeta_{n_j})) - \psi(\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{m_j}, \zeta_{n_j})) + \gamma\varphi(\mathfrak{N}(\zeta_{m_j}, \zeta_{n_j}))\big\}$$

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$$\leq \vartheta(lim_{j\to\infty}\sup\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{m_j},\zeta_{n_j})) - lim_{j\to\infty}\inf\psi(\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{m_j},\zeta_{n_j})) + 0$$

$$\leq \vartheta(s\epsilon) - \psi(lim_{\to\infty}\inf\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{m_j},\zeta_{n_j})).$$

From this, it follows that $\psi(lim\inf_{j\to\infty}\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{m_j},\zeta_{n_j})) = 0$.
Using the properties of ψ. we get $lim\inf_{j\to\infty}\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{m_j},\zeta_{n_j}) = 0$. which contradicts equation (14). This contradiction guarantees that $\{\eta_n\}$ is a b-cauchy sequence in $\mathfrak{M}_{\mathfrak{b}}$. By completeness of $\mathfrak{M}_{\mathfrak{b}}$ $\exists$ $\kappa \in \mathfrak{M}_{\mathfrak{b}}$ s.t.

$$lim_{n\to\infty}\eta_{2n+1} = lim_{n\to\infty}\mathfrak{F}\zeta_{2n+1} = lim_{n\to\infty}\mathfrak{T}\zeta_{2n+2} = \kappa \tag{16}$$

$$lim_{n\to\infty}\eta_{2n+2} = lim_{n\to\infty}\mathfrak{G}\zeta_{2n+2} = lim_{n\to\infty}\mathfrak{P}\zeta_{2n+3} = \kappa$$

Since $\mathfrak{P}(\mathfrak{M}_{\mathfrak{b}})$ is closed, we have $\kappa \in \mathfrak{P}(\mathfrak{M}_{\mathfrak{b}})$. So,we can choose a point $\mu \in \mathfrak{M}_{\mathfrak{b}}$ s.t. $\kappa = \mathfrak{S}\mu$ and rewrite the equation (16) as

$$lim_{n\to\infty}\eta_{2n+1} = lim_{n\to\infty}\mathfrak{F}\zeta_{2n+1} = lim_{n\to\infty}\mathfrak{T}\zeta_{2n+2} = \mathfrak{P}\mu = \kappa \tag{17}$$

$$lim_{n\to\infty}\eta_{2n+2} = lim_{n\to\infty}\mathfrak{G}\zeta_{2n+2} = lim_{n\to\infty}\mathfrak{P}\zeta_{2n+3} = \mathfrak{P}\mu = \kappa$$

We aim to assure that μ is a CP of $\mathfrak{F}$ and $\mathfrak{P}$ that is $\mathfrak{F}\mu = \mathfrak{P}\mu = \kappa$. Assume, that $\mathfrak{F}\mu \neq \kappa$. Condition (iv) ensure that $\alpha(\mathfrak{P}\zeta_n,\kappa)$=$\alpha(\mathfrak{P}\zeta_n,\mathfrak{P}\mu) \geq s$ for $n \in \mathbb{N}_0$.
Since $\mathfrak{P}\mu = \kappa$, put $\zeta = \mu, \eta = \zeta_{2n+2}$ in (1)

$$\vartheta(sd(\mathfrak{F}\mu,\mathfrak{G}\zeta_{2n+2})) \leq \vartheta(\alpha(\mathfrak{P}\mu,\mathfrak{T}\zeta_{2n+2})d(\mathfrak{F}\mu,\mathfrak{G}\zeta_{2n+2})) \leq \vartheta(\mathfrak{M}_{\mathfrak{b}_1}(\mu,\zeta_{2n+2}) - \psi(\mathfrak{M}_{\mathfrak{b}_1}(\mu,\zeta_{2n+2}) + \gamma\varphi(\mathfrak{N}(\mu,\zeta_{2n+2}) \tag{18}$$

$$\mathfrak{M}_{\mathfrak{b}_1}(\mu,\zeta_{2n+2}) = max\Big\{d(\mathfrak{P}\mu,\mathfrak{T}\zeta_{2n+2}), d(\mathfrak{P}\mu,\mathfrak{F}\mu), d(\mathfrak{T}\zeta_{2n+2},\mathfrak{G}\zeta_{2n+2}),$$

$$\frac{1}{2s}[d(\mathfrak{P}\mu,\mathfrak{G}\zeta_{2n+2}) + d(\mathfrak{T}\zeta_{2n+2},\mathfrak{F}\mu)], \frac{d(\mathfrak{T}\zeta_{2n+2},\mathfrak{G}\zeta_{2n+2})[1+d(\mathfrak{P}\mu,\mathfrak{F}\mu)]}{1+d(\mathfrak{P}\mu,\mathfrak{T}\zeta_{2n+2})}\Big\}$$

Approaching as $n \to \infty$

$$lim_{n\to\infty}\mathfrak{M}_{\mathfrak{b}_1}(\mu,\zeta_{2n+2}) = max\Big\{d(\kappa,\kappa), d(\kappa,\mathfrak{F}\mu), d(\kappa,\kappa), \frac{1}{2s}$$

$$[d(\kappa,\kappa) + d(\kappa,\mathfrak{F}\mu)], \frac{d(\kappa,\kappa)[1+d(\kappa,\mathfrak{F}\mu)]}{1+d(\kappa,\kappa)}\Big\}$$

$$lim_{n\to\infty}\mathfrak{M}_{\mathfrak{b}_1}(\mu,\zeta_{2n+2}) = max\left\{0, d(\kappa,\mathfrak{F}\mu), 0, \frac{1}{2s}[0+d(\kappa,\mathfrak{F}\mu)], \frac{[1+d(\kappa,\mathfrak{F}\mu)]}{1+0}\right\}$$

$$lim_{n\to\infty}\mathfrak{M}_{\mathfrak{b}_1}(\mu,\zeta_{2n+2}) \leq d(\kappa,\mathfrak{F}(\mu)) \tag{19}$$

$$lim_{n\to\infty}\mathfrak{N}(\mu,\zeta_{2n+2}) = min\left\{d(\mathfrak{S}\mu,\mathfrak{F}\mu), d(\mathfrak{T}\zeta_{2n+2},\mathfrak{G}\zeta_{2n+2}), d(\mathfrak{P}\mu,\mathfrak{G}\zeta_{2n+2}), d(\mathfrak{T}\zeta_{2n+2},\mathfrak{F}\mu)\right\}$$

$$lim_{n\to\infty}\mathfrak{N}(\mu,\zeta_{2n+2}) = min\left\{d(\kappa,\mathfrak{F}\mu), d(\kappa,\kappa), d(\kappa,\kappa), d(\kappa,\mathfrak{F}\mu)\right\}$$

$$= min\left\{d(\kappa,\mathfrak{F}\mu), 0, 0, d(\kappa,\mathfrak{F}\mu)\right\} = 0 \tag{20}$$

Approaching sup in equation (18) together with equation (19),(20) and lemma 2.2, we derive

$$\begin{aligned}
\vartheta(d(\mathfrak{F}\mu,\kappa)) &\leq \vartheta(s lim_{n\to\infty}\sup d(\mathfrak{F}\mu,\mathfrak{G}\zeta_{2n+2})) \\
&\leq lim_{n\to\infty}\sup \vartheta(\alpha(\mathfrak{P}\mu,\mathfrak{T}\zeta_{2n+2})d(\mathfrak{F}\mu,\mathfrak{G}\zeta_{2n+2})) \\
&\leq lim_{n\to\infty}\sup\left\{\vartheta(\mathfrak{M}_{\mathfrak{b}_1}(\mu,\zeta_{2n+2})) - \psi(\mathfrak{M}_{\mathfrak{b}_1}(\mu,\zeta_{2n+2})) + \gamma\varphi(\mathfrak{N}(\mu,\zeta_{2n+2}))\right\} \\
&\leq lim_{n\to\infty}\sup \vartheta(\mathfrak{M}_{\mathfrak{b}_1}(\mu,\zeta_{2n+2})) - lim_{n\to\infty}\inf \psi(\mathfrak{M}_{\mathfrak{b}_1}(\mu,\zeta_{2n+2})) + lim_{n\to\infty}\sup \gamma\varphi(\mathfrak{N}(\mu,\zeta_{2n+2})) \\
&\leq \vartheta(d(\kappa,\mathfrak{F}\mu)) - \psi(d(\kappa,\mathfrak{F}\mu)).
\end{aligned}$$

From this it follow that $\psi(d(\kappa,\mathfrak{F}\mu)) = 0$. By property of ψ,we have $d(\kappa,\mathfrak{F}\mu) = 0$. Therefore,$\mathfrak{F}\mu = \kappa$. That is, μ is a CP of $\mathfrak{F}$ and $\mathfrak{P}$.

$$\implies \mathfrak{F}\mu = \mathfrak{S}\mu = \kappa$$

Since $\mathfrak{F},\mathfrak{P}$ are weakly compatible.

$$\implies \mathfrak{F}\mathfrak{P}\mu = \mathfrak{S}\mathfrak{F}\mu$$

$$\implies \mathfrak{F}\kappa = \mathfrak{P}\kappa$$

Since $\mathfrak{F} \subset \mathfrak{T} \implies \kappa = \mathfrak{F}\mu \in \mathfrak{T}\ \exists\ q \in \mathfrak{M}_{\mathfrak{b}}$ s.t. $\mathfrak{T}q = \kappa$. We claim that $\mathfrak{G}q = \kappa$ put $\zeta = \zeta_{2n+1}$, $\eta = q$ in (1)

$$\begin{aligned}
\vartheta(sd(\mathfrak{F}\zeta_{2n+1},\mathfrak{G}q))) &\leq \vartheta(\alpha(\mathfrak{P}\zeta_{2n+1},\mathfrak{T}q))d(\mathfrak{F}\zeta_{2n+1},Gq) \\
&\leq \vartheta(\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{2n+1},q)) - \psi(\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{2n+1},q)) + \gamma\varphi(\mathfrak{N}(\zeta_{2n+1},q))
\end{aligned} \tag{21}$$

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$$\vartheta(\alpha(\mathfrak{P}\zeta_{2n+1},\mathfrak{T}q))d(\kappa,\mathfrak{G}q)\leq\vartheta(\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{2n+1},q))-\psi(\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{2n+1},q))+\gamma\varphi(\mathfrak{N}(\zeta_{2n+1},q))$$

$$\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{2n+1},q)=max\Big\{d(\mathfrak{P}\zeta_{2n+1},\mathfrak{T}q),d(\mathfrak{P}\zeta_{2n+1},\mathfrak{F}\zeta_{2n+1}),d(\mathfrak{T}q,\mathfrak{G}q),$$
$$\frac{1}{2s}[d(\mathfrak{P}\zeta_{2n+1},\mathfrak{G}q)+d(Tq,F\zeta_{2n+1})],\frac{d(\mathfrak{T}q,\mathfrak{G}q)[1+d(\mathfrak{P}\zeta_{2n+1},\mathfrak{F}\zeta_{2n+1})]}{1+d(\mathfrak{P}\zeta_{2n+1},\mathfrak{T}q)}\Big\}$$

Approaching as $n\to\infty$

$$lim_{n\to\infty}\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{2n+1},q)=max\Big\{d(\kappa,\kappa),d(\kappa,\kappa),d(z,\mathfrak{G}q),\frac{1}{2s}[d(z,\mathfrak{G}q)+d(\kappa,\kappa)],$$
$$\frac{d(\kappa,\mathfrak{G}q)[1+d(\kappa,\kappa)]}{1+d(\kappa,\kappa)}\Big\}$$

$$lim_{n\to\infty}\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{2n+1},q)=max\left\{0,0,d(\kappa,\mathfrak{G}q),\frac{1}{2s}[d(\kappa,\mathfrak{G}q)+0],d(\kappa,\mathfrak{G}q)\right\}$$

$$lim_{n\to\infty}\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{2n+1},q)\leq d(\kappa,\mathfrak{G}q) \tag{22}$$

$$lim_{n\to\infty}\mathfrak{N}(\zeta_{2n+1},q)=min\left\{d(\mathfrak{P}\zeta_{2n+1},\mathfrak{F}\zeta_{2n+1}),d(\mathfrak{T}q,\mathfrak{G}q),d(\mathfrak{P}\zeta_{2n+1},\mathfrak{G}q),d(\mathfrak{T}q,\mathfrak{F}\zeta_{2n+1})\right\}$$

$$lim_{n\to\infty}\mathfrak{N}(\zeta_{2n+1},q)=min.\left\{d(\kappa,\kappa),d(\kappa,\mathfrak{G}q),d(\kappa,\mathfrak{G}q),d(\kappa,\kappa)\right\}=0 \tag{23}$$

Approaching sup in equation (21) together with equation (22),(23) and lemma 2.2, we derive

$$\vartheta(d(\kappa,\mathfrak{G}q)\leq\vartheta(slim_{n\to\infty}\sup d(\mathfrak{F}\zeta_{2n+1},\mathfrak{G}q))$$
$$\leq lim_{n\to\infty}\sup\vartheta(\alpha(\mathfrak{P}\zeta_{2n+1},\mathfrak{T}q)d(\mathfrak{F}\zeta_{2n+1},\mathfrak{G}q)$$
$$\leq lim_{n\to\infty}\sup\left\{\vartheta(\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{2n+1},q)-\psi\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{2n+1},q)+\gamma\varphi(\mathfrak{N}(\zeta_{2n+1},q)\right\}$$
$$\leq lim_{n\to\infty}\sup\vartheta(\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{2n+1},q)-lim_{n\to\infty}\inf\psi(\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{2n+1},q)+lim_{n\to\infty}\sup\gamma\varphi(\mathfrak{N}(\zeta_{2n+1},q)$$
$$\leq\vartheta(d(\kappa,\mathfrak{G}q))-\psi(d(\kappa,\mathfrak{G}q)).$$

From this it follow that $\psi(d(\kappa,\mathfrak{G}q))=0$. By property of ψ,we have $d(\kappa,\mathfrak{G}q)=0$. Therefore,$\mathfrak{G}q=\kappa$. That is, q is a CP of $\mathfrak{G}$ and $\mathfrak{T}$.

$$\implies\kappa=\mathfrak{G}q\implies\mathfrak{T}q=\mathfrak{G}q$$

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The pair $(\mathfrak{G}, \mathfrak{T})$ is weakly compatible,we have

$$\mathfrak{T}\mathfrak{G}q = \mathfrak{G}\mathfrak{T}q \implies \mathfrak{T}\kappa = \mathfrak{G}\kappa$$

Now, we claim that $\mathfrak{F}\kappa = \kappa$. Put $\zeta = \kappa$, $\eta = \zeta_{2n+2}$ in (1)

$$\vartheta(sd(\mathfrak{F}\kappa, \mathfrak{G}\zeta_{2n+2}) \leq \vartheta(\alpha(\mathfrak{P}\kappa, \mathfrak{T}\zeta_{2n+2})d(\mathfrak{F}\kappa, \mathfrak{G}\zeta_{2n+2})) \leq \vartheta(\mathfrak{M}_{\mathfrak{b}_1}(\kappa, \zeta_{2n+2}) - \psi(\mathfrak{M}_{\mathfrak{b}_1}(\kappa, \zeta_{2n+2})$$
$$+\gamma\varphi(\mathfrak{N}(\kappa, \zeta_{2n+2})$$

$$\mathfrak{M}_{\mathfrak{b}_1}(\kappa, \zeta_{2n+2}) = max\Big\{d(\mathfrak{P}\kappa, \mathfrak{T}\zeta_{2n+2}), d(\mathfrak{P}\kappa, \mathfrak{F}\kappa), d(\mathfrak{T}\zeta_{2n+2}, \mathfrak{G}_{2n+2}),$$
$$\frac{1}{2s}[d(\mathfrak{P}\kappa, \mathfrak{G}\zeta_{2n+2}) + d(\mathfrak{T}\zeta_{2n+2}, \mathfrak{F}\kappa)], \frac{d(\mathfrak{T}\zeta_{2n+2}, \mathfrak{G}\zeta_{2n+2})[1 + d(\mathfrak{P}\kappa, \mathfrak{F}\kappa)]}{1 + d(\mathfrak{P}\kappa, \mathfrak{T}\zeta_{2n+2})}\Big\}$$

Approaching as $n \to \infty$

$$lim_{n\to\infty}\mathfrak{M}_{\mathfrak{b}_1}(\kappa, \zeta_{2n+2}) = max\Big\{d(\mathfrak{P}\kappa, \kappa), d(\mathfrak{P}\kappa, \mathfrak{F}\kappa), d(\kappa, \kappa),$$
$$\frac{1}{2s}[d(\mathfrak{P}\kappa, \kappa) + d(\kappa, \mathfrak{F}\kappa)], \frac{d(\kappa, \kappa)[1 + d(\mathfrak{P}\kappa, \mathfrak{F}\kappa)]}{1 + d(\mathfrak{P}\kappa, \kappa)}\Big\} = d(\mathfrak{F}\kappa, \kappa)$$
$$\mathfrak{N}(\kappa, \zeta_{2n+2}) = min\,\{d(\mathfrak{P}\kappa, \mathfrak{F}\kappa), d(\mathfrak{T}\zeta_{2n+2}, \mathfrak{G}\zeta_{2n+2}), d(\mathfrak{P}\kappa, \mathfrak{G}\zeta_{2n+2}), d(\mathfrak{T}\zeta_{2n+2}, \mathfrak{F}\kappa)\}$$
$$\mathfrak{N}(\kappa, \zeta_{2n+2}) = min\,\{d(\mathfrak{P}\kappa, \mathfrak{F}\kappa), d(\kappa, \kappa), d(\mathfrak{P}\kappa, \mathfrak{G}\zeta_{2n+2}), d(\kappa, \mathfrak{F}\kappa)\} = 0$$

Approaching sup.

$$\vartheta(d(\mathfrak{F}\kappa, \kappa)) \leq \vartheta(sd(\mathfrak{F}\kappa, \kappa)) \leq \vartheta(d(\mathfrak{F}\kappa, \kappa)) - \psi(d(\mathfrak{F}\kappa, \kappa))$$

$$\implies \mathfrak{F}\kappa = \kappa$$

Now, we claim that $\mathfrak{G}\kappa = \kappa$. Put $\zeta = \zeta_{2n+1}$, $\eta = \kappa$ in (1)

$$\vartheta(\alpha(\mathfrak{P}\zeta_{2n+1}, \mathfrak{T}\kappa)d(\mathfrak{F}\zeta_{2n+1}, \mathfrak{G}\kappa) \leq \vartheta(\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{2n+1}, \kappa) - \psi(\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{2n+1}, \kappa) + \gamma\varphi(\mathfrak{N}(\zeta_{2n+1}, \kappa)$$

$$\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{2n+1}, \kappa) = max\Big\{d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{T}\kappa), d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{F}\zeta_{2n+1}), d(\mathfrak{T}\kappa, \mathfrak{G}\kappa),$$

$$\frac{1}{2s}[d(\mathfrak{P}\zeta_{2n+1},\mathfrak{G}\kappa)+d(\mathfrak{T}\kappa,\mathfrak{F}\zeta_{2n+1})],\frac{d(\mathfrak{T}\kappa,\mathfrak{G}\kappa)[1+d(\mathfrak{P}\zeta_{2n+1},\mathfrak{F}\zeta_{2n+1})]}{1+d(\mathfrak{P}\zeta_{2n+1},\mathfrak{T}\kappa)}\Big\}$$

Approaching as $n\to\infty$

$$lim_{n\to\infty}\mathfrak{M}_{\mathfrak{b}_1}(\zeta_{2n+1},\kappa)=max\Big\{d(\kappa,\mathfrak{T}\kappa),d(\kappa,\kappa),d(\mathfrak{T}\kappa,\mathfrak{G}\kappa),$$

$$\frac{1}{2s}[d(\mathfrak{P}\kappa,\mathfrak{G}\kappa)+d(\mathfrak{T}\kappa,\kappa)],\frac{d(\mathfrak{T}\kappa,\mathfrak{G}\kappa)[1+d(\kappa,\kappa)]}{1+d(\kappa+\mathfrak{T}\kappa)}\Big\}$$

$$=d(\mathfrak{G}\kappa,\kappa)$$

$$\mathfrak{N}(\zeta_{2n+1},\kappa)=min\left\{d(\mathfrak{P}\zeta_{2n+1},\mathfrak{F}\kappa),d(\mathfrak{T}\kappa,\mathfrak{G}\kappa),d(\mathfrak{P}\zeta_{2n+1},\mathfrak{G}\kappa),d(\mathfrak{T}\kappa,\mathfrak{F}\zeta_{2n+1})\right\}$$

$$\mathfrak{N}(\zeta_{2n+1},\kappa)=min\left\{d(\kappa,\mathfrak{F}\kappa),d(\mathfrak{T}\kappa,\mathfrak{G}\kappa),d(\kappa,\mathfrak{G}\kappa),d(\kappa,\kappa)\right\}$$

$$\mathfrak{N}(\zeta_{2n+1},\kappa)=min\left\{0,d(\mathfrak{T}\kappa,\mathfrak{G}\kappa),d(\kappa,\mathfrak{G}\kappa),0\right\}=0$$

Approaching sup.

$$\vartheta(d(\kappa,\mathfrak{G}\kappa))\leq\vartheta(sd(\kappa,\mathfrak{G}\kappa))\leq\vartheta(d(\kappa,\mathfrak{G}\kappa))-\psi(d(\kappa,\mathfrak{G}\kappa))$$

$$\implies\mathfrak{G}\kappa=\kappa.$$

We deduce that $\mathfrak{S}\kappa=\mathfrak{T}\kappa=\mathfrak{F}\kappa=\mathfrak{G}\kappa=\kappa$ and hence κ is a CFP.
To prove uniqueness, assume that ϖ is another CFP of $\mathfrak{T},\mathfrak{S},\mathfrak{F}$ and $\mathfrak{G}$ such that $\kappa\neq\varpi$. Thus, we have

$$\mathfrak{F}\kappa=\mathfrak{G}\kappa=\mathfrak{P}\kappa=\mathfrak{T}\kappa=\kappa$$

$$\mathfrak{F}\varpi=\mathfrak{G}\varpi=\mathfrak{P}\varpi=\mathfrak{T}\varpi=\varpi$$

By means of (v), it holds that $\alpha(\mathfrak{P}\kappa,\mathfrak{T}\varpi)\geq s$. Implement equation (1) with $\zeta=\kappa$, $\eta=\varpi$

$$\vartheta(d(\kappa,\varpi))=\vartheta(d(\mathfrak{F}\kappa,\mathfrak{G}\varpi)))\leq\vartheta(sd(\mathfrak{F}\kappa,\mathfrak{G}\varpi))$$

$$\leq\vartheta(\alpha(\mathfrak{P}\kappa,\mathfrak{T}\varpi)d(\mathfrak{F}\kappa,\mathfrak{G}\varpi))\leq\vartheta(\mathfrak{M}_{\mathfrak{b}_1}(\kappa,\varpi))-\psi(\mathfrak{M}_{\mathfrak{b}_1}(\kappa,\varpi))+\gamma\varphi(\mathfrak{N}(\kappa,\varpi))\quad(24)$$

$$\mathfrak{M}_{\mathfrak{b}_1}(\kappa,\varpi)=max\Big\{d(\mathfrak{P}\kappa,\mathfrak{T}\varpi),d(\mathfrak{P}\kappa,\mathfrak{F}\kappa),d(\mathfrak{T}\varpi,\mathfrak{G}\varpi),\frac{1}{2s}[d(\mathfrak{P}\kappa,\mathfrak{G}\varpi)+d(\mathfrak{T}\varpi,\mathfrak{F}\kappa)],$$

$$\left.\frac{d(\mathfrak{T}\varpi,\mathfrak{G}\varpi)[1+d(\mathfrak{P}\kappa,\mathfrak{F}\kappa)]}{1+d(\mathfrak{P}\kappa,\mathfrak{T}\varpi)}\right\}$$

$$\mathfrak{M}_{\mathfrak{b}_1}(\kappa,\varpi)=max\left\{d(\kappa,\varpi),d(\kappa,\kappa),d(\varpi,\varpi),\frac{1}{2s}[d(\kappa,\varpi)+d(\varpi,\kappa)],\frac{d(\varpi,\varpi)[1+d(\kappa,\kappa)]}{1+d(\kappa,\varpi)}\right\}$$
$$=d(\kappa,\varpi)$$

$\mathfrak{N}(\kappa,\varpi)=min\left\{d(\mathfrak{P}\kappa,\mathfrak{F}\kappa),d(\mathfrak{T}\varpi,\mathfrak{G}\varpi),d(\mathfrak{P}\kappa,\mathfrak{G}\varpi),d(\mathfrak{T}\varpi,\mathfrak{F}\kappa)\right\}$
$\mathfrak{N}(\kappa,\varpi)=min\left\{d(\kappa,\kappa),d(\varpi,\varpi),d(\kappa,\varpi),d(\varpi,\kappa)\right\}=0$
Consequently, it follows from equation (24) that

$$\vartheta(d(\kappa,\varpi))\leq\vartheta(\mathfrak{M}_{\mathfrak{b}_1}(\kappa,\varpi))-\psi(\mathfrak{M}_{\mathfrak{b}_1}(\kappa,\varpi))+\gamma\varphi(\mathfrak{N}(\kappa,\varpi))$$

$$\leq\vartheta(d(\kappa,\varpi))-\psi(d(\kappa,\varpi)).$$

which,using property of ϑ and ψ, implies that $d(\kappa,\varpi)=0$. Which contradicts our assumption $\varpi\neq\kappa$. Therefore, $\varpi=\kappa$. This proves uniqueness.

Theorem 2.11 Let $(\mathfrak{M}_{\mathfrak{b}},d,s)$ be a b-CMS with $s\geq1$ and let $\alpha{:}\mathfrak{M}_{\mathfrak{b}}\times\mathfrak{M}_{\mathfrak{b}}\to\mathbb{R}^+$ be a function. Suppose $\mathfrak{T},\mathfrak{P},\mathfrak{F},\mathfrak{G}:\mathfrak{M}_{\mathfrak{b}}\to\mathfrak{M}_{\mathfrak{b}}$ are mappings s.t. $(\mathfrak{F},\mathfrak{G})$ is α- admissible w.r.t. $(\mathfrak{P},\mathfrak{T})$ and gratifying the following axioms:

(i) the pair $(\mathfrak{T},\mathfrak{P}),(\mathfrak{F},\mathfrak{G})$ is a broad rational type (ϑ,ψ,φ) weak contraction mapping with $\mathfrak{M}_{\mathfrak{b}_i}=\mathfrak{M}_{\mathfrak{b}_2}$;
(ii) $\mathfrak{F}(\mathfrak{M}_{\mathfrak{b}})\subseteq\mathfrak{T}(\mathfrak{M}_{\mathfrak{b}})$, $\mathfrak{G}(\mathfrak{M}_{\mathfrak{b}})\subseteq\mathfrak{P}(\mathfrak{M}_{\mathfrak{b}})$ and $\mathfrak{T}(\mathfrak{M}_{\mathfrak{b}})$ or $\mathfrak{P}(\mathfrak{M}_{\mathfrak{b}})$ is a closed subspace of $\mathfrak{M}_{\mathfrak{b}}$;
(iii) $\exists\ \zeta_1\in\mathfrak{M}_{\mathfrak{b}}$ s.t. $\alpha(\mathfrak{P}\zeta_1,\mathfrak{F}\zeta_1)\geq s$ and $\alpha(\mathfrak{F}\zeta_1,\mathfrak{P}\zeta_1)\geq s$;
(iv) for any non-decreasing sequence $\{\mathfrak{P}\zeta_n\}$ in $\mathfrak{M}_{\mathfrak{b}}$ with $\mathfrak{P}\zeta_n\to\mathfrak{P}\zeta$ in $\mathfrak{P}(\mathfrak{M}_{\mathfrak{b}})$ as $n\to\infty$ and $\alpha(\mathfrak{P}\zeta_n,\mathfrak{P}\zeta_{n+1})\geq s\ \forall\ n\in\mathbb{N}_0$, we have $\alpha(\mathfrak{P}\zeta_n,\mathfrak{P}\zeta)\ \forall\ n\in\mathbb{N}_0$;
(v) $\alpha(\mathfrak{P}\kappa,\mathfrak{T}\varpi)\geq s\ \forall\ \varpi,\kappa\in\mathfrak{C}P(\mathfrak{G},\mathfrak{T})$.

Then $\mathfrak{T},\mathfrak{P},\mathfrak{F},\mathfrak{G}$ admit unique CP in $\mathfrak{M}_{\mathfrak{b}}$. Moreover, $(\mathfrak{T},\mathfrak{P})$ and $(\mathfrak{F},\mathfrak{G})$ have unique CFP in $\mathfrak{M}_{\mathfrak{b}}$ provided that $(\mathfrak{F},\mathfrak{P})$ and $(\mathfrak{G},\mathfrak{T})$ are weakly compatible.
Proof: From (iii) $\exists\ \zeta_1\in\mathfrak{M}_{\mathfrak{b}}$ s.t. $\alpha(\mathfrak{P}\zeta_1,\mathfrak{F}\zeta_1)\geq s$ and $\alpha(\mathfrak{F}\zeta_1,\mathfrak{P}\zeta_1)\geq s$. Since $\mathfrak{F}(\mathfrak{M}_{\mathfrak{b}})\subseteq\mathfrak{T}(\mathfrak{M}_{\mathfrak{b}})$, $\mathfrak{G}(\mathfrak{M}_{\mathfrak{b}})\subseteq\mathfrak{P}(\mathfrak{M}_{\mathfrak{b}})$, $\exists$ sequences $\{\zeta_n\}$ and $\{\eta_n\}$ in $\mathfrak{M}_{\mathfrak{b}}$ s.t.

$$\eta_{2n+1}=\mathfrak{F}\zeta_{2n+1}=\mathfrak{T}\zeta_{2n+2}\text{ and }\eta_{2n+2}=\mathfrak{G}\zeta_{2n+2}=\mathfrak{P}\zeta_{2n+3},n\in\mathbb{N}_0 \tag{25}$$

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Now we have
$\alpha(\mathfrak{P}\zeta_1, \mathfrak{F}\zeta_1) \geq s \implies \alpha(\mathfrak{P}\zeta_1, \mathfrak{T}\zeta_2) \geq s,$

$$\implies \alpha(\mathfrak{F}\zeta_1, \mathfrak{G}\zeta_2) \geq s, i.e. \alpha(\eta_1, \eta_2) \geq s$$

$$\implies \alpha(\mathfrak{T}\zeta_2, \mathfrak{P}\zeta_3) \geq s,$$

$$\implies \alpha(\mathfrak{G}\zeta_2, \mathfrak{F}\zeta_3) \geq s, i.e. \alpha(\eta_2, \eta_3) \geq s$$

$$\implies \alpha(\mathfrak{P}\zeta_3, \mathfrak{T}\zeta_4) \geq s$$

$$\implies \alpha(\mathfrak{F}\zeta_3, \mathfrak{G}\zeta_4) \geq s, i.e. \alpha(\eta_3, \eta_4) \geq s$$

Continuing in this way,we have

$$\alpha(\eta_n, \eta_{n+1}) \geq s \, \forall \, n \in \mathbb{N} \tag{26}$$

Similarly by using $\alpha(\mathfrak{F}\zeta_1, \mathfrak{P}\zeta_1) \geq s$ we can claim that

$$\alpha(\eta_{n+1}, \eta_n) \geq s \, \forall \, n \in \mathbb{N}$$

From (26), we have if $\eta_{2n+1} = \eta_{2n+2}$ for some $n \in \mathbb{N}$ then $\mathfrak{T}\zeta_{2n+2} = \eta_{2n+1} = \eta_{2n+2} = \mathfrak{F}\zeta_{2n+2}$ and from (26)
If $\eta_{2n+2} = \eta_{2n+3}$ for some $n \in \mathbb{N}$ then $\mathfrak{P}\zeta_{2n+3} = \eta_{2n+2} = \eta_{2n+3} = \mathfrak{G}\zeta_{2n+3}$ $\forall$ $n \in \mathbb{N}_0$, that $(\mathfrak{P}, \mathfrak{G})$ and $(\mathfrak{T}, \mathfrak{F})$ have a CP at ζ_{2n+2} and ζ_{2n+3} and we have finished the proof so we assume that $\eta_n \neq \eta_{n+1}$ $\forall$ $n \in \mathbb{N}_0$.
from (27), $\alpha(\mathfrak{P}\zeta_{2n+1}, \mathfrak{T}\zeta_{2n+2}) = \alpha(\eta_{2n}, \eta_{2n+1}) \geq s$
from (1) we have
Case I. If n is odd i.e., n = 2n+1, n = 0,1,2,.... then we get

$$\vartheta(sd(\eta_{2n+1}, \eta_{2n+2}) = \vartheta(sd(\mathfrak{F}\zeta_{2n+1}, \mathfrak{G}\zeta_{2n+2})$$

$$\leq \vartheta(\alpha(\mathfrak{P}\zeta_{2n+1}, \mathfrak{T}\zeta_{2n+2})d(\mathfrak{F}\zeta_{2n+1}, \mathfrak{G}\zeta_{2n+2})$$

$$\leq \vartheta(\mathfrak{M}_{\mathfrak{b}_z}(\zeta_{2n+1}, \zeta_{2n+2})) - \psi(\mathfrak{M}_{\mathfrak{b}_z}(\zeta_{2n+1}, \zeta_{2n+2})) + \gamma\varphi(\mathfrak{N}(\zeta_{2n+1}, \zeta_{2n+2})) \tag{27}$$

where

$$\mathfrak{M}_{\mathfrak{b}_2}(\zeta_{2n+1}, \zeta_{2n+2}) = max\Big\{d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{T}\zeta_{2n+2}), d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{F}\zeta_{2n+1}), d(\mathfrak{T}\zeta_{2n+2}, \mathfrak{G}\zeta_{2n+2}),$$

$$\frac{d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{F}\zeta_{2n+1})d(\mathfrak{T}\zeta_{2n+2}, \mathfrak{F}\zeta_{2n+1}) + d(\mathfrak{T}\zeta_{2n+2}, \mathfrak{G}\zeta_{2n+2})d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{G}\zeta_{2n+2})}{1 + d(\mathfrak{T}\zeta_{2n+2}, \mathfrak{F}\zeta_{2n+1}) + d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{G}\zeta_{2n+2})},$$

$$\frac{d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{F}\zeta_{2n+1})d(\mathfrak{T}\zeta_{2n+2}, \mathfrak{F}\zeta_{2n+1}) + d(\mathfrak{T}\zeta_{2n+2}, \mathfrak{G}\zeta_{2n+2})d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{G}\zeta_{2n+2})}{1 + s(d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{F}\zeta_{2n+1}) + d(\mathfrak{T}\zeta_{2n+2}, \mathfrak{G}\zeta_{2n+2}))}\Big\},$$

$$\mathfrak{M}_{\mathfrak{b}_2}(\zeta_{2n+1}, \zeta_{2n+2}) = \Big\{(\eta_{2n}, \eta_{2n+1}), d(\eta_{2n}, \eta_{2n+1}), d(\mathfrak{T}\eta_{2n+1}, \mathfrak{G}\eta_{2n+2}),$$

$$\frac{d(\eta_{2n}, \eta_{2n+1})d(\eta_{2n+1}, \eta_{2n+1}) + d(\eta_{2n+1}, \eta_{2n+2})d(\eta_{2n}, \eta_{2n+2})}{1 + d(\eta_{2n}, \eta_{2n+2})}$$

$$\frac{d(\eta_{2n}, \eta_{2n+1})d(\eta_{2n+1}, \eta_{2n+1}) + d(\eta_{2n+1}, \eta_{2n+2})d(\eta_{2n}, \eta_{2n+2})}{1 + s(d(\eta_{2n}, \eta_{2n+1})) + d(\eta_{2n+1}, \eta_{2n+2})}\Big\},$$

$$\leq max\,\{d(\eta_{2n}, \eta_{2n+1}), d(\eta_{2n+1}, \eta_{2n+2})\}$$

$$\leq max\,\{d(\eta_{2n}, \eta_{2n+1}), d(\eta_{2n+1}, \eta_{2n+2})\} \tag{28}$$

$$\mathfrak{N}(\zeta_{2n+1}, \zeta_{2n+2}) = min\Big\{d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{F}\zeta_{2n+1}), d(T\zeta_{2n+2}, \mathfrak{G}\zeta_{2n+2}),$$

$$d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{G}\zeta_{2n+2}), d(\mathfrak{T}\zeta_{2n+2}, \mathfrak{F}\zeta_{2n+1})\Big\}$$

$$= min\,\{d(\eta_{2n}, \eta_{2n+1}), d(\eta_{2n+1}, \eta_{2n+2}), d(\eta_{2n}, \eta_{2n+2})d(\eta_{2n+1}, \eta_{2n+1})\} = 0 \tag{29}$$

If $d(\eta_{2n+1}, \eta_{2n+2}) > d(\eta_{2n}, \eta_{2n+1})$ for some $n \in \mathbb{N}$, then by equations (27)-(29), we have

$$\vartheta(d(\eta_{2n+1}, \eta_{2n+2}) \leq \vartheta(sd(\eta_{2n+1}, \eta_{2n+2})$$

$$\leq \vartheta(\mathfrak{M}_2(\zeta_{2n+1}, \zeta_{2n+2})) - \psi(\mathfrak{M}_{\mathfrak{b}_2}(\zeta_{2n+1}, \zeta_{2n+2})) + \gamma\varphi(0)$$

$$\leq \vartheta(\mathfrak{M}_{\mathfrak{b}_2}(\zeta_{2n+1}, \zeta_{2n+2}))$$

$$< \vartheta(d(\eta_{2n+1}, \eta_{2n+2})$$

which is contradicton. Thus from equation (28) we have

$$d(\eta_{2n+1}, \eta_{2n+2}) \leq \mathfrak{M}_{\mathfrak{b}_{\mathfrak{z}}}(\zeta_{2n+1}, \zeta_{2n+2}) \leq d(\eta_{2n}, \eta_{2n+1}). \tag{30}$$

Case II. If n is even then similar case I. Hence $\{d(\eta_n, \eta_{n+1})\}$ is decreasing sequence. Let $lim_{n\to\infty} d(\eta_n, \eta_{n+1}) = \lambda,$ for some $\lambda \geq 0$.
From equation (30) it follows that

$$lim_{n\to\infty} d(\eta_{2n+1}, \eta_{2n+2}) = lim_{n\to\infty} \mathfrak{M}_{\mathfrak{b}_{\mathfrak{z}}}(\zeta_{2n+1}, \zeta_{2n+2}) = \lambda. \tag{31}$$

We claim $\lambda = 0$. Suppose that $\lambda > 0$. Approaching as $n \to \infty$ of the inequality in Equation (27) together with Equation (29) and (31), we have

$$= lim_{n\to\infty} \vartheta(sd(\eta_{2n+1}, \eta_{2n+2})$$

$$\begin{aligned}\vartheta(\lambda) \leq \vartheta(s\lambda) = lim_{n\to\infty} \vartheta(sd(\eta_{2n+1}, \eta_{2n+2}) &\leq lim_{n\to\infty} \Big\{ \vartheta(\mathfrak{M}_{\mathfrak{b}_{\mathfrak{z}}}(\zeta_{2n+1}, \zeta_{2n+2})) \\ &- \psi(\mathfrak{M}_{\mathfrak{b}_{\mathfrak{z}}}(\zeta_{2n+1}, \zeta_{2n+2})) + \gamma\varphi(\mathfrak{N}(\zeta_{2n+1}, \zeta_{2n+2}) \Big\} \\ &\leq \vartheta(\lambda) - \psi(\lambda) + \gamma\varphi(0) \\ &< \vartheta(\lambda)\end{aligned}$$

which is a contradiction thus, $\lambda = 0$ and hence

$$lim_{n\to\infty} d(\eta_n, \eta_{n+1}) = 0 \tag{32}$$

Next, we claim that the sequence $\{\eta_n\}$ is b-cauchy in $\mathfrak{M}_{\mathfrak{b}}$. Suppose,that $\{\eta_n\}$ is not b-cauchy. By lemma 2.1, $\exists\ \epsilon > 0$ and two subsequences $\{\eta_{n_j}\}$ and $\{\eta_{m_j}\}$ of $\{\eta_n\}$ $n_j > m_j \geq j\ \forall\ j \in \mathbb{N}$ and $d(\eta_{m_j}, \eta_{n_j}) \geq \epsilon$, $d(\eta_{m_j}, \eta_{n_j-1}) < \epsilon$
Since $n_j > m_j\ \forall\ j \in \mathbb{N}$ using equation (26) we have $\alpha(\mathfrak{P}\zeta_{m_j}, \mathfrak{T}\zeta_{n_j}) \geq s$. Now, putting $\zeta = \zeta_{m_j}$ and $\eta = \zeta_{n_j}$ in equation (1), we derive

$$\begin{aligned}\vartheta(sd(\eta_{m_j}, \eta_{n_j})) &= \vartheta(sd(\mathfrak{F}\zeta_{m_j}, \mathfrak{G}\zeta_{n_j})) \\ &\leq \vartheta(\alpha(\mathfrak{P}\zeta_{m_j}, \mathfrak{T}\zeta_{n_j}) d(\mathfrak{F}\zeta_{m_j}, \mathfrak{G}\zeta_{n_j}))\end{aligned}$$

$$\leq \vartheta(\mathfrak{M}_{\mathfrak{b}_{\mathfrak{z}}}(\zeta_{m_j}, \zeta_{n_j})) - \psi(\mathfrak{M}_{\mathfrak{b}_{\mathfrak{z}}}(\zeta_{m_j}, x_{n_j})) + \gamma\varphi(\mathfrak{N}(\zeta_{m_j}, \zeta_{n_j})) \tag{33}$$

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where

$$\mathfrak{M}_{\mathfrak{b}_2}(\zeta_{m_j}, \zeta_{n_j}) = max\Big\{ d(\mathfrak{P}\zeta_{m_j}, \mathfrak{T}\zeta_{n_j}), d(\mathfrak{P}\zeta_{m_j}, \mathfrak{F}\zeta_{m_j}), d(\mathfrak{T}\zeta_{n_j}, \mathfrak{G}\zeta_{n_j}), \frac{d(\mathfrak{P}\zeta_{m_j}, \mathfrak{F}\zeta_{n_j})d(\mathfrak{T}\zeta_{n_j}, \mathfrak{F}\zeta_{m_j}) + d(\mathfrak{T}\zeta_{n_j}, \mathfrak{G}\zeta_{n_j})d(\mathfrak{P}\zeta_{m_j}, \mathfrak{G}\zeta_{n_j})}{1 + d(\mathfrak{T}\zeta_{n_j}, \mathfrak{F}\zeta_{m_j}) + d(\mathfrak{P}\zeta_{m_j}, \mathfrak{G}\zeta_{n_j})}, \frac{d(\mathfrak{P}\zeta_{m_j}, \mathfrak{F}\zeta_{m_j})d(\mathfrak{T}\zeta_{n_j}, \mathfrak{F}\zeta_{m_j}) + d(\mathfrak{T}\zeta_{n_j}, \mathfrak{G}\zeta_{n_j})d(\mathfrak{P}\zeta_{m_j}, \mathfrak{G}\zeta_{n_j})}{1 + s(d(\mathfrak{P}\zeta_{m_j}, \mathfrak{F}\zeta_{m_j}) + d(\mathfrak{T}\zeta_{n_j}, \mathfrak{G}\zeta_{n_j}))} \Big\}$$

$$= max\Big\{ d(\eta_{m_j-1}, \eta_{n_j-1}), d(\eta_{m_j-1}, \eta_{m_j}), d(\eta_{n_j-1}, y_{n_j}), \frac{d(\eta_{m_j-1}, \eta_{m_j})d(\eta_{n_j}, \eta_{m_j-1}) + d(\eta_{n_j-1}, \eta_{n_j})d(\eta_{m_j}, \eta_{n_j-1})}{1 + d(\eta_{n_j}, y_{m_j-1)} + d(\eta_{m_j}, \eta_{n_j-1})} \frac{d(\eta_{m_j}, \eta_{m_j-1})d(\eta_{n_j}, \eta_{m_j-1}) + d(\eta_{n_j}, \eta_{n_j-1})d(\eta_{m_j}, \eta_{n_j-1})}{1 + s[d(\eta_{m_j}, \eta_{m_j-1}) + d(\eta_{n_j}, \eta_{n_j-1})]} \Big\} \quad (34)$$

$$\mathfrak{N}(\zeta_{m_j}, \zeta_{n_j}) = min\left\{ d(\mathfrak{P}\zeta_{m_j}, \mathfrak{F}\zeta_{n_j}), d(\mathfrak{T}\zeta_{n_j}, \mathfrak{G}\zeta_{n_j}), d(\mathfrak{P}\zeta_{m_j}, \mathfrak{G}\zeta_{n_j}), d(\mathfrak{T}\zeta_{m_j}, \mathfrak{G}\zeta_{n_j}), d(\mathfrak{T}\zeta_{n_j}, \mathfrak{F}\zeta_{m_j}) \right\} \quad (35)$$

$$= min\left\{ d(\eta_{m_j-1}, \eta_{m_j}), d(\eta_{n_j-1}, \eta_{n_j}), d(\eta_{m_j}, \eta_{n_j-1}), d(\eta_{n_j}, \eta_{m_j-1}) \right\}$$

Now, approaching as $j \to \infty$ in equation (34) together with lemma 2.1 and using equation (32), we have

$$lim_{j\to\infty} \sup \mathfrak{M}_{\mathfrak{b}_2}(\zeta_{m_j}, x_{n_j}) \leq max\left\{ s\epsilon, 0, 0, 0, 0 \right\} = s\epsilon, \quad (36)$$

and

$$s\epsilon \geq lim_{j\to\infty} \inf \mathfrak{M}_{\mathfrak{b}_2}(\zeta_{m_j}, \zeta_{n_j}) \geq \frac{\epsilon}{s^2}. \quad (37)$$

Also,approaching as $j \to \infty$ in equation (35) together with lemma 2.1 and using equation (32), we have

$$0 \leq lim_{j\to\infty} \sup \mathfrak{N}(\zeta_{m_j}, \zeta_{n_j}) \leq min\left\{ 0, 0, \epsilon, s^2\epsilon \right\} = 0. \quad (38)$$

Approaching sup in equation (33) together with lemma 2.1 and using equation (36)

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and (38), we have

$$\vartheta(s\epsilon) \leq lim_{j\to\infty} \sup \vartheta(sd(\eta_{m_j}, \eta_{n_j}))$$

$$\leq lim_{j\to\infty} \sup \left\{\vartheta(\mathfrak{M}_{\mathfrak{b}_{\mathfrak{z}}}(\zeta_{m_j}, \zeta_{n_j})) - \psi(\mathfrak{M}_{\mathfrak{b}_{\mathfrak{z}}}(\zeta_{m_j}, \zeta_{n_j})) + \gamma\varphi(\mathfrak{N}(\zeta_{m_j}, \zeta_{n_j}))\right\}$$

$$\leq \vartheta(lim_{j\to\infty} \sup \mathfrak{M}_{\mathfrak{b}_{\mathfrak{z}}}(\zeta_{m_j}, \zeta_{n_j})) - lim_{j\to\infty} \inf \psi(\mathfrak{M}_{\mathfrak{b}_{\mathfrak{z}}}(\zeta_{m_j}, \zeta_{n_j})) + 0$$

$$\leq \vartheta(s\epsilon) - \psi(lim_{\to\infty} \inf \mathfrak{M}_{\mathfrak{b}_{\mathfrak{z}}}(\zeta_{m_j}, \zeta_{n_j})).$$

From this, it follows that $\psi(lim \inf_{j\to\infty} \mathfrak{M}_{\mathfrak{b}_{\mathfrak{z}}}(\zeta_{m_j}, \zeta_{n_j})) = 0$.
Using the properties of ψ. we get $lim \inf_{j\to\infty} \mathfrak{M}_{\mathfrak{b}_{\mathfrak{z}}}(\zeta_{m_j}, \zeta_{n_j}) = 0$. which contradicts equation (37). This contradiction guarantees that $\{\eta_n\}$ is a b-cauchy sequence in $\mathfrak{M}_{\mathfrak{b}}$. By completeness of $\mathfrak{M}_{\mathfrak{b}}$ $\exists$ $\kappa \in \mathfrak{M}_{\mathfrak{b}}$ s.t.

$$lim_{n\to\infty}\eta_{2n+1} = lim_{n\to\infty}\mathfrak{F}\zeta_{2n+1} = lim_{n\to\infty}\mathfrak{T}\zeta_{2n+2} = \kappa \tag{39}$$

$$lim_{n\to\infty}\eta_{2n+2} = lim_{n\to\infty}\mathfrak{G}\zeta_{2n+2} = lim_{n\to\infty}\mathfrak{P}\zeta_{2n+3} = \kappa$$

Since $\mathfrak{P}(\mathfrak{M}_{\mathfrak{b}})$ is closed,we have $\kappa \in \mathfrak{P}(\mathfrak{M}_{\mathfrak{b}})$. So,we can choose a $\mu \in \mathfrak{M}_{\mathfrak{b}}$ s.t. $\kappa = \mathfrak{S}\mu$. and rewrite the equation (39) as

$$lim_{n\to\infty}\eta_{2n+1} = lim_{n\to\infty}\mathfrak{F}\zeta_{2n+1} = lim_{n\to\infty}\mathfrak{T}\zeta_{2n+2} = \mathfrak{S}\mu = \kappa \tag{40}$$

$$lim_{n\to\infty}\eta_{2n+2} = lim_{n\to\infty}\mathfrak{G}\zeta_{2n+2} = lim_{n\to\infty}\mathfrak{P}\zeta_{2n+3} = \mathfrak{P}\mu = \kappa$$

We aim to assure that μ is a CP of $\mathfrak{F}$ and $\mathfrak{P}$ that is $\mathfrak{F}\mu = \mathfrak{P}\mu = \kappa$. Assume, that $\mathfrak{F}\mu \neq \kappa$. Condition (iv) ensure that $\alpha(\mathfrak{P}\zeta_n, \kappa)$=$\alpha(\mathfrak{P}\zeta_n, \mathfrak{P}\mu) \geq s$ for $n \in \mathbb{N}_0$. Since $\mathfrak{P}\mu = \kappa$, put $\zeta = \mu, \eta = \zeta_{2n+2}$ in (1)

$$\vartheta(sd(\mathfrak{F}\mu, \mathfrak{G}\zeta_{2n+2})) \leq \vartheta(\alpha(\mathfrak{P}\mu, \mathfrak{T}\zeta_{2n+2})d(\mathfrak{F}\mu, \mathfrak{G}\zeta_{2n+2})) \leq \vartheta(\mathfrak{M}_{\mathfrak{z}}(\mu, \zeta_{2n+2}) - \psi(\mathfrak{M}_{\mathfrak{z}}(\mu, \zeta_{2n+2}) \\ +\gamma\varphi(\mathfrak{N}(\mu, \zeta_{2n+2}) \tag{41}$$

$$\mathfrak{M}_{\mathfrak{b}_{\mathfrak{z}}}(\mu, \zeta_{2n+2}) = max\Big\{d(\mathfrak{P}\mu, \mathfrak{T}\zeta_{2n+2}), d(\mathfrak{P}\mu, \mathfrak{F}\mu), d(\mathfrak{T}\zeta_{2n+2}, \mathfrak{G}\zeta_{2n+2}),$$

$$\frac{d(\mathfrak{P}\mu, \mathfrak{F}\mu)d(\mathfrak{T}\zeta_{2n+2}, \mathfrak{F}\mu) + d(\mathfrak{T}\zeta_{2n+2}, \mathfrak{G}\zeta_{2n+2})d(\mathfrak{P}\mu, \mathfrak{G}\zeta_{2n+2})}{1 + d(\mathfrak{T}\zeta_{2n+2}, \mathfrak{F}\mu) + d(\mathfrak{P}\mu, \mathfrak{G}\zeta_{2n+2})},$$

$$\frac{d(\mathfrak{S}\mu, \mathfrak{F}\mu)d(\mathfrak{T}\zeta_{2n+2}, \mathfrak{F}\mu) + d(\mathfrak{T}\zeta_{2n+2}, \mathfrak{G}\zeta_{2n+2})d(\mathfrak{P}\mu, \mathfrak{G}\zeta_{2n+2})}{1 + sd(\mathfrak{P}\mu, \mathfrak{F}\mu) + d(\mathfrak{T}\zeta_{2n+2}, \mathfrak{G}\zeta_{2n+2})}\Big\},$$

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Approaching as $n \to \infty$

$$lim_{n\to\infty}\mathfrak{M}_{\mathfrak{b}_z}(\mu,\zeta_{2n+2}) = max\Big\{d(\kappa,\kappa), d(\kappa,\mathfrak{F}\mu), d(\kappa,\kappa), \frac{d(\kappa,\mathfrak{F}\mu)d(\kappa,\mathfrak{F}\mu)+d(\kappa,\kappa)d(\kappa,\kappa)}{1+d(\kappa,\mathfrak{F}\mu)+d(\kappa,\kappa)},$$

$$\frac{d(\kappa,\mathfrak{F}\mu)d(\kappa,\mathfrak{F}\mu)+d(\kappa,\kappa)d(\kappa,\kappa)}{1+sd(\kappa,\mathfrak{F}\mu)+d(\kappa,\kappa)}\Big\}$$

$$lim_{n\to\infty}\mathfrak{M}_{\mathfrak{b}_z}(\mu,\zeta_{2n+2}) = max\Big\{0, d(\kappa,\mathfrak{F}\mu), 0, \frac{d(\kappa,\mathfrak{F}\mu)d(\kappa,\mathfrak{F}\mu)+0}{1+d(\kappa,\mathfrak{F}\mu)+0}, \frac{d(\kappa,\mathfrak{F}\mu)d(\kappa,\mathfrak{F}\mu)+0}{1+sd(\kappa,\mathfrak{F}\mu)+0}\Big\}$$

$$lim_{n\to\infty}\mathfrak{M}_{\mathfrak{b}_z}(\mu,\zeta_{2n+2}) \le d(\kappa,\mathfrak{F}(\mu)) \tag{42}$$

$$lim_{n\to\infty}\mathfrak{N}(\mu,\zeta_{2n+2}) = min\left\{d(\mathfrak{P}\mu,\mathfrak{F}\mu), d(\mathfrak{T}\zeta_{2n+2},\mathfrak{G}\zeta_{2n+2}), d(\mathfrak{P}\mu,\mathfrak{G}\zeta_{2n+2}), d(\mathfrak{T}\zeta_{2n+2},\mathfrak{F}\mu)\right\}$$

$$lim_{n\to\infty}\mathfrak{N}(\mu,\zeta_{2n+2}) = min.\left\{d(\kappa,\mathfrak{F}\mu), d(\kappa,\kappa), d(\kappa,\kappa), d(\kappa,\mathfrak{F}\mu)\right\}$$

$$= min\left\{d(\kappa,\mathfrak{F}\mu), 0, 0, d(\kappa,\mathfrak{F}\mu)\right\} = 0 \tag{43}$$

Approaching sup in equation (41) together with equation (42),(43) and lemma 2, we derive

$$\vartheta(d(\mathfrak{F}\mu,\kappa)) \le \vartheta(slim_{n\to\infty}\sup d(\mathfrak{F}\mu,\mathfrak{G}\zeta_{2n+2}))$$

$$\le lim_{n\to\infty}\sup \vartheta(\alpha(\mathfrak{P}\mu,\mathfrak{T}\zeta_{2n+2})d(\mathfrak{F}\mu,\mathfrak{G}\zeta_{2n+2}))$$

$$\le lim_{n\to\infty}\sup\left\{\vartheta(\mathfrak{M}_{\mathfrak{b}_z}(\mu,\zeta_{2n+2}) - \psi(\mathfrak{M}_{\mathfrak{b}_z}(\mu,\zeta_{2n+2}) + \gamma\varphi(\mathfrak{N}(\mu,\zeta_{2n+2})\right\}$$

$$\le lim_{n\to\infty}\sup\vartheta(\mathfrak{M}_{\mathfrak{b}_z}(\mu,\zeta_{2n+2}) - lim_{n\to\infty}\inf\psi(\mathfrak{M}_{\mathfrak{b}_z}(\mu,\zeta_{2n+2}) + lim_{n\to\infty}\sup\gamma\varphi(\mathfrak{N}(\mu,\zeta_{2n+2})$$

$$\le \vartheta(d(\kappa,\mathfrak{F}\mu)) - \psi(d(\kappa,\mathfrak{F}\mu)).$$

From this it follow that $\psi(d(\kappa,\mathfrak{F}\mu)) = 0$. By property of ψ,we have $d(\kappa,\mathfrak{F}\mu) = 0$. Therefore,$\mathfrak{F}\mu = \kappa$. Thus, μ is a CP of $\mathfrak{F}$ and $\mathfrak{P}$, i.e. $\mathfrak{F}\mu = \mathfrak{P}\mu = \kappa$.

Since the mappings $\mathfrak{F}$ and $\mathfrak{P}$ are weakly compatible, We derive $\mathfrak{F}\mathfrak{P}\mu = \mathfrak{P}\mathfrak{F}\mu$, i.e., $\mathfrak{F}\kappa = \mathfrak{P}\kappa$ Since $\mathfrak{F} \subset \mathfrak{T} \implies \kappa = \mathfrak{F}\mu \in \mathfrak{T}\ \exists\ q \in \mathfrak{M}_{\mathfrak{b}}$ s.t. $\mathfrak{T}q = \kappa$. We claim that

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$\mathfrak{G}q = \kappa$ put $\zeta = \zeta_{2n+1}$, $\eta = q$ in (1)

$$\vartheta(sd(\mathfrak{F}\zeta_{2n+1}, \mathfrak{G}q) \leq \vartheta(\alpha(\mathfrak{P}\zeta_{2n+1}, \mathfrak{T}q)d(\mathfrak{F}\zeta_{2n+1}, \mathfrak{G}q)) \leq \vartheta(\mathfrak{M}_{\mathfrak{b}_2}(\zeta_{2n+1}, q) - \psi(\mathfrak{M}_{\mathfrak{b}_2}(\zeta_{2n+1}, q) + \gamma\varphi(\mathfrak{N}(\zeta_{2n+1}, q) \tag{44}$$

$$\vartheta(\alpha(\mathfrak{P}\zeta_{2n+1}, \mathfrak{T}q)d(\kappa, \mathfrak{G}q)) \leq \vartheta(\mathfrak{M}_{\mathfrak{b}_2}(\zeta_{2n+1}, q) - \psi(\mathfrak{M}_{\mathfrak{b}_2}(\zeta_{2n+1}, q) + \gamma\varphi(\mathfrak{N}(\zeta_{2n+1}, q)$$

$$\mathfrak{M}_{\mathfrak{b}_2}(\zeta_{2n+1}, q) = max\Big\{d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{T}q), d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{F}\zeta_{2n+1}), d(\mathfrak{T}q, \mathfrak{G}q), \frac{d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{F}\zeta_{2n+1})d(\mathfrak{T}q, \mathfrak{F}\zeta_{2n+1}) + d(\mathfrak{T}q, \mathfrak{G}q)d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{G}q)}{1 + d(\mathfrak{T}q, \mathfrak{F}\zeta_{2n+1}) + d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{G}q)}, \frac{d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{F}\zeta_{2n+1})d(\mathfrak{T}q, \mathfrak{F}\zeta_{2n+1}) + d(\mathfrak{T}q, \mathfrak{G}q)d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{G}q)}{1 + sd(\mathfrak{P}\zeta_{2n+1}, \mathfrak{F}\zeta_{2n+1}) + d(\mathfrak{T}q, \mathfrak{G}q)}\Big\},$$

Approaching as $n \to \infty$

$$lim_{n\to\infty}\mathfrak{M}_{\mathfrak{b}_2}(\zeta_{2n+1}, q) = max\Big\{d(\kappa, \kappa), d(\kappa, \kappa), d(\kappa, \mathfrak{G}q), \frac{d(\kappa, \kappa)d(\kappa, \kappa) + d(\kappa, \mathfrak{G}q)d(\kappa, \mathfrak{G}q)}{1 + d(\kappa, \kappa) + d(\kappa, \mathfrak{G}q)}, \frac{d(\kappa, \kappa)d(\kappa, \kappa) + d(\kappa, \mathfrak{G}q)d(\kappa, \mathfrak{G}q)}{1 + sd(\kappa, \kappa) + d(\kappa, \mathfrak{G}q)}\Big\}$$

$$lim_{n\to\infty}\mathfrak{M}_{\mathfrak{b}_2}(\zeta_{2n+1}, q) = max\Big\{0, 0, d(\kappa, \mathfrak{G}q), \frac{d(\kappa, \mathfrak{G}q)d(\kappa, \mathfrak{G}q) + 0}{1 + d(\kappa, \mathfrak{G}q) + 0}, \frac{d(\kappa, \mathfrak{G}q)d(\kappa, \mathfrak{G}q) + 0}{1 + d(\kappa, \mathfrak{G}q)}\Big\}$$

$$lim_{n\to\infty}\mathfrak{M}_{\mathfrak{b}_2}(\zeta_{2n+1}, q) \leq d(\kappa, \mathfrak{G}q) \tag{45}$$

$$lim_{n\to\infty}\mathfrak{N}(\zeta_{2n+1}, q) = min\left\{d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{F}\zeta_{2n+1}), d(\mathfrak{T}q, \mathfrak{G}q), d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{G}q), d(\mathfrak{T}q, \mathfrak{F}\zeta_{2n+1})\right\}$$

$$lim_{n\to\infty}\mathfrak{N}(\zeta_{2n+1}, q) = min\left\{d(\kappa, \kappa), d(\kappa, \mathfrak{G}q), d(\kappa, \mathfrak{G}q), d(\kappa, \kappa)\right\} = 0 \tag{46}$$

Approaching sup in equation (44) together with equation (45),(46) and lemma 2.2, we derive

$$\vartheta(d(\kappa, \mathfrak{G}q) \leq \vartheta(slim_{n\to\infty} \sup d(\mathfrak{F}\zeta_{2n+1}, \mathfrak{G}q))$$

$$\leq lim_{n\to\infty} \sup \vartheta(\alpha(\mathfrak{P}\zeta_{2n+1}, \mathfrak{T}q)d(\mathfrak{F}\zeta_{2n+1}, \mathfrak{G}q)$$

$$\leq lim_{n\to\infty} \sup\left\{\vartheta(\mathfrak{M}_{\mathfrak{b}_2}(\zeta_{2n+1}, q) - \psi(\mathfrak{M}_{\mathfrak{b}_2}(\zeta_{2n+1}, q) + \gamma\varphi(\mathfrak{N}(\zeta_{2n+1}, q)\right\}$$

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$$\leq lim_{n\to\infty}\sup\vartheta(\mathfrak{M}_{\mathfrak{b}_z}(\zeta_{2n+1},q) - lim_{n\to\infty}\inf\psi(\mathfrak{M}_{\mathfrak{b}_z}(\zeta_{2n+1},q) + lim_{n\to\infty}\sup\gamma\varphi(\mathfrak{N}(\zeta_{2n+1},q)$$

$$\leq \vartheta(d(\kappa,\mathfrak{G}q)) - \psi(d(\kappa,\mathfrak{G}q)).$$

From this it follow that $\psi(d(\kappa,\mathfrak{G}q)) = 0$. By property of ψ,we have $d(\kappa,\mathfrak{G}q) = 0$. Therefore,$\mathfrak{G}q = \kappa$. That is, q is a CP of $\mathfrak{G}$ and $\mathfrak{T}$.

$$\implies \kappa = \mathfrak{G}q \implies \mathfrak{T}q = \mathfrak{G}q$$

The pair $(\mathfrak{G},\mathfrak{T})$is weakly compatible,we have

$$\mathfrak{T}\mathfrak{G}q = \mathfrak{G}\mathfrak{T}q \implies \mathfrak{T}\kappa = \mathfrak{G}\kappa$$

Now, we claim that $\mathfrak{F}\kappa = \kappa$. Put $\zeta = \kappa$, $\eta = \zeta_{2n+2}$ in (1)

$$\vartheta(\alpha(\mathfrak{S}\kappa,\mathfrak{T}\zeta_{2n+2})d(\mathfrak{F}\kappa,\mathfrak{G}\zeta_{2n+2})) \leq \vartheta(\mathfrak{M}_{\mathfrak{b}_z}(\kappa,\zeta_{2n+2}) - \psi(\mathfrak{M}_{\mathfrak{b}_z}(\kappa,\zeta_{2n+2}) + \gamma\varphi(\mathfrak{N}(\kappa,\zeta_{2n+2})$$

$$\begin{aligned}\mathfrak{M}_{\mathfrak{b}_z}(\kappa,\zeta_{2n+2}) = max\Big\{&d(\mathfrak{P}\kappa,\mathfrak{T}\zeta_{2n+2}), d(\mathfrak{P}\kappa,\mathfrak{F}\kappa), d(\mathfrak{T}\zeta_{2n+2},\mathfrak{G}\zeta_{2n+2}),\\ &\frac{d(\mathfrak{P}\kappa,\mathfrak{F}\kappa)d(\mathfrak{T}\zeta_{2n+2},\mathfrak{F}\kappa) + d(\mathfrak{T}\zeta_{2n+2},\mathfrak{G}\zeta_{2n+2})d(\mathfrak{P}\kappa,\mathfrak{G}\zeta_{2n+2})}{1 + d(\mathfrak{T}\zeta_{2n+2},\mathfrak{F}\kappa) + d(\mathfrak{P}\kappa,\mathfrak{G}\zeta_{2n+2})},\\ &\frac{d(\mathfrak{P}\kappa,\mathfrak{F}\kappa)d(\mathfrak{T}\zeta_{2n+2},\mathfrak{F}\kappa) + d(\mathfrak{T}\zeta_{2n+2},\mathfrak{G}\zeta_{2n+2})d(\mathfrak{P}\kappa,\mathfrak{G}\zeta_{2n+2})}{1 + sd(\mathfrak{P}\kappa,\mathfrak{F}\kappa) + d(\mathfrak{T}\zeta_{2n+2},\mathfrak{G}\zeta_{2n+2})}\Big\},\end{aligned}$$

Approaching as $n \to \infty$

$$lim_{n\to\infty}\mathfrak{M}_{\mathfrak{b}_z}(\kappa,\zeta_{2n+2}) = max\Big\{d(\mathfrak{P}\kappa,\kappa), d(\mathfrak{P}\kappa,\kappa), d(\kappa,\kappa), \frac{d(\mathfrak{P}\kappa,\kappa)d(\kappa,\kappa) + d(\kappa,\kappa)d(\mathfrak{P}\kappa,\kappa)}{1 + d(\mathfrak{P}\kappa,\kappa) + d(\kappa,\kappa)},$$

$$\frac{d(\mathfrak{P}\kappa,\kappa)d(\kappa,\kappa) + d(\kappa,\kappa)d(\mathfrak{P}\kappa,\kappa)}{1 + sd(\mathfrak{P}\kappa,\kappa) + d(\kappa,\kappa)}\Big\} = d(\mathfrak{F}\kappa,\kappa)$$

$$\mathfrak{N}(\kappa,\zeta_{2n+2}) = min\left\{d(\mathfrak{P}\kappa,\mathfrak{F}\kappa), d(\mathfrak{T}\zeta_{2n+2},\mathfrak{G}\zeta_{2n+2}), d(\mathfrak{P}\kappa,\mathfrak{G}\zeta_{2n+2}), d(\mathfrak{T}\zeta_{2n+2},\mathfrak{F}\kappa)\right\}$$

$$\mathfrak{N}(\kappa,\zeta_{2n+2}) = min\left\{d(\mathfrak{P}\kappa,\mathfrak{F}\kappa), d(\kappa,\kappa), d(\mathfrak{P}\kappa,\mathfrak{G}\zeta_{2n+2}), d(\kappa,\mathfrak{F}\kappa)\right\} = 0$$

Approaching sup.

$$\vartheta(d(\mathfrak{F}\kappa,\kappa)) \leq \vartheta(sd(\mathfrak{F}\kappa,\kappa)) \leq \vartheta(d(\mathfrak{F}\kappa,\kappa)) - \psi(d(\mathfrak{F}\kappa,\kappa))$$

$$\implies \mathfrak{F}\kappa = \kappa$$

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Now, we claim that $\mathfrak{G}\kappa = \kappa$. Put $\zeta = \zeta_{2n+1}$, $\eta = \zeta$ in (1)

$$\vartheta(\alpha(\mathfrak{P}\zeta_{2n+1}, \mathfrak{T}\kappa)d(\mathfrak{F}\zeta_{2n+1}, \mathfrak{G}\kappa) \leq \vartheta(\mathfrak{M}_{\mathfrak{b}_2}(\zeta_{2n+1}, \kappa) - \psi(\mathfrak{M}_{\mathfrak{b}_2}(\zeta_{2n+1}, \kappa) + \gamma\varphi(\mathfrak{N}(\zeta_{2n+1}, \kappa)$$

$$\mathfrak{M}_{\mathfrak{b}_2}(\zeta_{2n+1}, \kappa) = max\Big\{d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{T}\kappa), d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{F}\zeta_{2n+1}), d(\mathfrak{T}\kappa, \mathfrak{G}\kappa),$$
$$\frac{d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{F}\zeta_{2n+1})d(\mathfrak{T}\kappa, \mathfrak{F}\zeta_{2n+1}) + d(\mathfrak{T}\kappa, \mathfrak{G}\kappa)d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{G}\kappa)}{1 + d(\mathfrak{T}\kappa, \mathfrak{F}\zeta_{2n+1}) + d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{G}\kappa)},$$
$$\frac{d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{F}\zeta_{2n+1})d(\mathfrak{T}\kappa, \mathfrak{F}\zeta_{2n+1}) + d(\mathfrak{T}\kappa, \mathfrak{G}\kappa)d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{G}\kappa)}{1 + sd(\mathfrak{P}\zeta_{2n+1}, \mathfrak{F}\zeta_{2n+1}) + d(\mathfrak{T}\kappa, \mathfrak{G}\kappa)}\Big\},$$

Approaching as $n \to \infty$

$$lim_{n\to\infty}\mathfrak{M}_{\mathfrak{b}_2}(\zeta_{2n+1}, \kappa) = max\Big\{d(\kappa, \mathfrak{T}\kappa), d(\kappa, \kappa), d(\mathfrak{T}\kappa, \mathfrak{G}\kappa), \frac{d(\kappa, \kappa)d(\mathfrak{T}\kappa, \kappa) + d(\mathfrak{T}\kappa, \mathfrak{G}\kappa)d(\kappa, \mathfrak{G}\kappa)}{1 + d(\mathfrak{T}\kappa, \kappa) + d(\kappa, \mathfrak{G}\kappa)},$$
$$\frac{d(\kappa, \kappa)d(\mathfrak{T}\kappa, \kappa) + d(\mathfrak{T}\kappa, \mathfrak{G}\kappa)d(\mathfrak{P}\kappa, \mathfrak{G}\kappa)}{1 + sd(\kappa, \kappa) + d(\mathfrak{T}\kappa, \mathfrak{G}\kappa)}\Big\} = d(\mathfrak{G}\kappa, \kappa).$$

$$\mathfrak{N}(\zeta_{2n+1}, \kappa) = min\left\{d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{F}\kappa), d(\mathfrak{T}\kappa, \mathfrak{G}\kappa), d(\mathfrak{P}\zeta_{2n+1}, \mathfrak{G}\kappa), d(\mathfrak{T}\kappa, \mathfrak{F}\zeta_{2n+1})\right\}$$
$$\mathfrak{N}(\zeta_{2n+1}, \kappa) = min\left\{d(\kappa, \mathfrak{F}\kappa), d(\mathfrak{T}\kappa, \mathfrak{G}\kappa), d(\kappa, \mathfrak{G}\kappa), d(\kappa, \kappa)\right\}$$
$$\mathfrak{N}(\zeta_{2n+1}, \kappa) = min\left\{0, d(\mathfrak{T}\kappa, \mathfrak{G}\kappa), d(\kappa, \mathfrak{G}\kappa), 0\right\} = 0.$$

Approaching sup.

$$\vartheta(d(\kappa, \mathfrak{G}\kappa)) \leq \vartheta(sd(\kappa, \mathfrak{G}\kappa)) \leq \vartheta(d(\kappa, \mathfrak{G}\kappa)) - \psi(d(\kappa, \mathfrak{G}\kappa))$$

$$\implies \mathfrak{G}\kappa = \kappa.$$

We deduce that $\mathfrak{P}\kappa = \mathfrak{T}\kappa = \mathfrak{F}\kappa = \mathfrak{G}\kappa = \kappa$ and hence κ is a CFP. Uniqueness follows easily from condition (1).

Example 2.12 Let $\mathfrak{M}_{\mathfrak{b}} = [0, 1]$ equipped with b-metric $d(\zeta, \eta) = |\zeta - \eta|^2 \ \forall \ \zeta, \eta \in \mathfrak{M}_{\mathfrak{b}}$. Consider the mapping $\mathfrak{F}, \mathfrak{G}, \mathfrak{P}, \mathfrak{T} : \mathfrak{M}_{\mathfrak{b}} \to \mathfrak{M}_{\mathfrak{b}}$ be defined by

$$\mathfrak{F}(\zeta) = \frac{\zeta}{8} \, \forall \, \zeta \in [0, 1]$$

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$$\mathfrak{G}(\zeta) = \begin{cases} 0, \zeta \in [0, \frac{1}{2}) \\ \frac{1}{16}, \zeta \in [\frac{1}{2}, 1] \end{cases}$$

$$\mathfrak{P}(\zeta) = \begin{cases} \frac{3x}{2}, \zeta \in [0, \frac{1}{2}) \\ \zeta, \zeta \in [\frac{1}{2}, 1] \end{cases}$$

$$\mathfrak{T}(\zeta) = \begin{cases} 2\zeta, \zeta \in [0, \frac{1}{2}) \\ \zeta, \zeta \in [\frac{1}{2}, 1] \end{cases}$$

Let $\alpha : \mathfrak{M}_\mathfrak{b} \times \mathfrak{M}_\mathfrak{b} \to \mathbb{R}^+$

$$\alpha(\zeta, \eta) = \begin{cases} 1, \zeta \in \mathfrak{M}_\mathfrak{b}, \eta = 1 \\ 2, otherwise \end{cases}$$

Let the function $\vartheta, \psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$, respectively, be given by

$$\vartheta(\xi) = 11\xi$$

$$\psi(\xi) = \begin{cases} 0, \xi = 0 \\ \frac{1}{8} + \frac{1}{8}\xi, 0 < \xi \leq 1; \\ 2, \xi > 1. \end{cases} \quad \text{and } \varphi(\xi) = \frac{\xi}{2}.$$

We assure that all the axioms of theorem 3.1 are gratified and $(\mathfrak{F}, \mathfrak{G})$ and $(\mathfrak{T}, \mathfrak{P})$ have a unique CFP in $\mathfrak{M}_\mathfrak{b}$.

Corollary 2.13 Let $(\mathfrak{M}_\mathfrak{b}, d, s)$ be a b-CMS with $s \geq 1$ and $\mathfrak{T}, \mathfrak{P}, \mathfrak{F}, \mathfrak{G} : \mathfrak{M}_\mathfrak{b} \to \mathfrak{M}_\mathfrak{b}$ be mappings s.t. $\mathfrak{F}(\mathfrak{M}_\mathfrak{b}) \subseteq \mathfrak{T}(\mathfrak{M}_\mathfrak{b})$, $\mathfrak{G}(\mathfrak{M}_\mathfrak{b}) \subseteq \mathfrak{P}(\mathfrak{M}_\mathfrak{b})$ and $\mathfrak{T}(\mathfrak{M}_\mathfrak{b})$ or $\mathfrak{P}(\mathfrak{M}_\mathfrak{b})$ is a closed subspace of $\mathfrak{M}_\mathfrak{b}$. Suppose that the pair $(\mathfrak{P}, \mathfrak{T})$ and gratifies the contractive axiom

$$sd(\mathfrak{F}\zeta, \mathfrak{G}\eta) \leq \vartheta(\mathfrak{M}_{\mathfrak{b}_\mathfrak{i}}(\zeta, \eta)) - \psi(\mathfrak{M}_{\mathfrak{b}_\mathfrak{i}}(\zeta, \eta)) + \gamma\varphi(\mathfrak{N}(\zeta, \eta)), \tag{47}$$

where $\mathfrak{M}_\mathfrak{i}$, for i=1,2, as defined above. Then $\mathfrak{T}, \mathfrak{P}, \mathfrak{F}, \mathfrak{G}$ have a CP in $\mathfrak{M}_\mathfrak{b}$. If, in addition, $\alpha(\mathfrak{P}\kappa, \mathfrak{T}\varpi) \geq s$ for any $\varpi, \kappa \in \mathfrak{C}P(\mathfrak{P}, \mathfrak{T})$ then the CP of $\mathfrak{P}$ and $\mathfrak{T}$ is

unique. Moreover,if the pair $(\mathfrak{T}, \mathfrak{P})$ and $(\mathfrak{F}, \mathfrak{G})$ is weakly compatible, then $(\mathfrak{T}, \mathfrak{P})$ and $(\mathfrak{F}, \mathfrak{G})$ have unique CFP in $\mathfrak{M}_{\mathfrak{b}}$.
Proof: Put $\varphi(\xi) = \xi \ \forall \ \xi \in \mathbb{R}^+$ and $\alpha(\zeta, \eta) = s \ \forall \ \zeta, \eta \in \mathfrak{M}_{\mathfrak{b}}$. and follow the steps in the proof of Theorems 3.1 and 3.2 we derive the desired result.
Putting $\mathfrak{T} = \mathfrak{P}$ and $\mathfrak{G} = \mathfrak{F}$ in theorems 3.1 and 3.2 we derive the theorems 3 and 4 of Kalo et al.[2]

Corollary 2.14 Let $(\mathfrak{M}_{\mathfrak{b}}, d, s)$ be a b-CMS with $s \geq 1$ and let α:$\mathfrak{M}_{\mathfrak{b}} \times \mathfrak{M}_{\mathfrak{b}} \to \mathbb{R}^+$ be a function. Suppose $\mathfrak{F}, \mathfrak{P} : \mathfrak{M}_{\mathfrak{b}} \to \mathfrak{M}_{\mathfrak{b}}$ are two self-mappings. Assume that $\mathfrak{F}$ is α- admissible, $\alpha(\mathfrak{F}\zeta_0, \mathfrak{P}\zeta_0) \geq s$ for certain $\zeta_0 \in \mathfrak{M}_{\mathfrak{b}}$ and $\exists \ \vartheta \in \vartheta, \psi \in \Psi, \varphi \in \Phi$ s.t. $\forall \ \zeta, \eta \in \mathfrak{M}_{\mathfrak{b}}$

$$\vartheta(\alpha(\mathfrak{P}\zeta, \mathfrak{P}\eta)d(\mathfrak{F}\zeta, \mathfrak{F}\eta)) \leq \vartheta({\mathfrak{M}_{\mathfrak{b}_i}}^*(\zeta, \eta)) - \psi({\mathfrak{M}_{\mathfrak{b}_i}}^*(\zeta, \eta)) + \gamma\varphi(\mathfrak{N}^*(\zeta, \eta)) \text{ where } i = 1, 2. \tag{48}$$

Then $\mathfrak{F}$ and $\mathfrak{P}$ has a FP in $\mathfrak{M}_{\mathfrak{b}}$ if $\{\zeta_n\}$ is a sequence in $\mathfrak{M}_{\mathfrak{b}}$ s.t. $\zeta_n \to \zeta \in \mathfrak{M}_{\mathfrak{b}}$ as $n \to \infty$ and $\alpha(\zeta_n, \zeta_{n+1}) \geq s \ \forall \ n \in \mathbb{N}$, then $\alpha(\zeta_n, \zeta) \geq s \ \forall \ n \in \mathbb{N}$. Moreover, $\mathfrak{F}$ and $\mathfrak{P}$ has unique CFP in $\mathfrak{M}_{\mathfrak{b}}$ provided that $\alpha(\mu, \varpi) \geq s$ for any $\mu, \varpi \in \mathfrak{F}P(\mathfrak{F}, \mathfrak{P})$.
Proof: Putting $\gamma = 0$ and $\alpha(\zeta, \varpi) = s$ in Corollary 3.5, we derive the following FP result.

Corollary 2.15 Let $(\mathfrak{M}_{\mathfrak{b}}, d, s)$ be a b-CMS with $s \geq 1$ and $\mathfrak{F}, \mathfrak{P} : \mathfrak{M}_{\mathfrak{b}} \to \mathfrak{M}_{\mathfrak{b}}$ are two self-mappings. Suppose that $\exists \ \vartheta \in \vartheta, \psi \in \Psi$ s.t. $\forall \ \zeta, \eta \in \mathfrak{M}_{\mathfrak{b}}$

$$\vartheta(sd(\mathfrak{F}\zeta, \mathfrak{G}\eta)) \leq \vartheta(\mathfrak{M}_{\mathfrak{b}_i}(\zeta, \eta)) - \psi(\mathfrak{M}_{\mathfrak{b}_i}(\zeta, \eta)). \tag{49}$$

Then $\mathfrak{F}$ and $\mathfrak{G}$ has unique FP in $\mathfrak{M}_{\mathfrak{b}}$.

Application

Let $\varpi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function. The Caputo derivative of function ϖ of order $\lambda > 0$ is given by

$$^c\mathfrak{D}^\lambda \varpi(t) := \frac{1}{\Gamma(n-\lambda)} \int_0^1 (t-\tau)^{n-\lambda-1} \varpi^n(\tau) d\tau, n-1 < \lambda \leq n; n = [\lambda] + 1,$$

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where $[\lambda]$ denote the integer part of the positive real number λ and Γ function given by

$$\Gamma(q) = \int_0^{\infty} t^{q-1}e^{-t}dt.$$

$$\text{Set } \mathfrak{G}(\mathfrak{J}) := \{h|h \text{ is a real continuous functions on } \mathfrak{J} = [0,1]\}.$$

Then, $\mathfrak{G}(\mathfrak{J})$ forms a Banach space with the norm $||h||_{\infty} = sup_{\tau \in j}|h(\tau)|$. Denote by $\mathfrak{G}(\mathfrak{J}, \mathbb{R})$ the set of all real-valued continuous functions from $j = [0,1]$. Consider the nonlinear fractional order BVP

$$\begin{cases} {}^{c}\mathfrak{D}^{\lambda}\mu(t) = \mathfrak{F}(t, \mu(t)), t \in \mathfrak{J} = [0,1], 0 < \lambda < 1, \\ \mu(0) = 0 = \mu(1). \end{cases} \tag{50}$$

where ${}^{c}\mathfrak{D}^{\lambda}$ is the Caputo derivative of order λ and $\mathfrak{F} : \mathfrak{J} \times \mathbb{R} \to \mathbb{R}$ is continuous and $\mu \in \mathfrak{G}(\mathfrak{J}, \mathbb{R})$.
The Green's function $\mathfrak{G} : [0,1] \to [0,1]$ associated with equation(51) is given by

$$\begin{cases} t(1-\tau)^{\lambda-1} - (t-\tau)^{\lambda-1}, 0 \leq \tau \leq t \leq 1; \\ \frac{t(1-\tau)^{1-\lambda}}{\Gamma(\lambda)}, 0 \leq t \leq \tau \leq 1 \end{cases} \tag{51}$$

The equivalent I.E. corresponding to equation (51) is given by

$$\mu(t) = \int_0^1 \mathfrak{G}(t,\tau)\mathfrak{F}_1(t, \mu(\tau))d\tau, \forall\, t \in \mathfrak{J}.$$
$$\mu(t) = \int_0^1 \mathfrak{G}(t,\tau)\mathfrak{F}_2(t, \mu(\tau))d\tau, \forall\, t \in \mathfrak{J}. \tag{52}$$

Let $\mathfrak{M}_{\mathfrak{b}} = \mathfrak{G}(\mathfrak{J}, \mathbb{R})$. Let $d : \mathfrak{M}_{\mathfrak{b}} \times \mathfrak{M}_{\mathfrak{b}} \to \mathbb{R}^{+}$ be defined by

$$d(\mu, v) = ||\mu - v||_{\infty}^{2} = \sup_{t \in J} |\mu(t) - v(t)|^{2}, \forall\, \mu, v \in \mathfrak{M}_{\mathfrak{b}} \tag{53}$$

Theorem 2.16 Suppose that the following axiom hold:

(i) $\mathfrak{T}_1, \mathfrak{T}_2 : J \times \mathbb{R} \to \mathbb{R}$ is continuous function;

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(ii) the integral $\mathfrak{S}, \mathfrak{T} : \mathfrak{M}_{\mathfrak{b}} \to \mathfrak{M}_{\mathfrak{b}}$ is defined by

$$\mathfrak{S}\mu(t) = \int_0^1 \mathfrak{G}(t,\tau)\mathfrak{T}_1(t,\mu(\tau))d\tau, \forall\, \mu \in \mathfrak{M}_{\mathfrak{b}}, t \in \mathfrak{J} \tag{54}$$

$$\mathfrak{T}\mu(t) = \int_0^1 \mathfrak{G}(t,\tau)\mathfrak{T}_2(t,v(\tau))d\tau, \forall\, v \in \mathfrak{M}_{\mathfrak{b}}, t \in \mathfrak{J} \tag{55}$$

(iii) the function $\mathfrak{G} : [0,1] \times [0,1] \to \mathbb{R}^+$ is continuous and gratify

$$sup_{t\in[0,1]} \int_0^1 |\mathfrak{G}(t,\tau)|d\tau \leq 1$$

(iv) $\exists$ a constant $k \in (0,1)$ s.t. $\forall\, t, \tau \in [0,1]$

$$|\mathfrak{T}_1(t,\mu(\tau)) - \mathfrak{T}_2(t,v(\tau))| \leq \sqrt{\frac{1-k}{2s}}|\mu(\tau) - v(\tau)|.$$

The the BVP (51) has a solution in $\mathfrak{M}_{\mathfrak{b}}$.

Proof: Clearly, $(\mathfrak{M}_{\mathfrak{b}}, d, s)$ is a b-CMS with $s = 2$. It is true that $\mu^* \in \mathfrak{M}_{\mathfrak{b}}$ is a solution of equation of equation (51) iff if $\mu^* \in \mathfrak{M}_{\mathfrak{b}}$ is a solution of the non linear I.E. (55) and (56). Equivalently, $\mu^* \in \mathfrak{M}_{\mathfrak{b}}$ is a solution of equation (51) iff $\mu^* \in \mathfrak{M}_{\mathfrak{b}}$ is a FP of the integral operator $(\mathfrak{P}, \mathfrak{T})$ defined in equation (55) and (56). using $(ii) - (iv)$, $\forall$ $t \in [0,1]$, we have

$$sd(\mathfrak{S}\mu(t), \mathfrak{T}v(t)) = s \sup_{t\in[0,1]} |\mathfrak{S}\mu(t) - \mathfrak{T}v(t)|^2$$

$$= s \sup_{t\in[0,1]} |\int_0^1 \mathfrak{G}(t,\tau)(\mathfrak{T}_1(t,\mu(\tau)) - (\mathfrak{T}_2(t,v(\tau))d\tau|^2$$

$$\leq s \sup_{t\in[0,1]} (\int_0^1 \mathfrak{G}(t,\tau)|d\tau|)^2 (\int_0^1 |\mathfrak{T}_1(t,\mu(\tau)) - (\mathfrak{T}_2(t,v(\tau))d\tau|)^2$$

$$\leq 2 \sup_{t\in[0,1]} \int_0^1 |\mathfrak{G}(t,\tau)|^2 d\tau \int_0^1 (\frac{1-k}{2s})|\mu(\tau) - v(\tau)|^2 d\tau \tag{56}$$

$$\leq (\frac{1-k}{2})|\mu(t) - v(t)|^2$$
$$\leq (\frac{1-k}{2})\mathfrak{M_1}^*(\mu(t), v(t)).$$

Define the mapping $\vartheta, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\vartheta(\xi) = \xi$ and $\psi(\xi) = \frac{k}{4}\xi$ for $\xi \in \mathbb{R}^+$. Applying equation (50) together with equation (57), we have

$$\vartheta(sd(\mathfrak{S}\mu(t), \mathfrak{T}v(t))) = \vartheta(s \sup_{t\in[0,1]} |\mathfrak{S}\mu(t) - \mathfrak{T}v(t)|^2)$$
$$\leq \vartheta((\frac{1-k}{2})\mathfrak{M_1}^*(\mu(t), v(t)))$$
$$= (\frac{1-k}{2})\mathfrak{M_1}^*(\mu(t), v(t))$$
$$\leq \vartheta(\mathfrak{M_1}^*(\mu(t), v(t))) - \psi(\mathfrak{M_1}^*(\mu(t), v(t))).$$

$\forall$ $\mu, v \in \mathfrak{M_b}$ and $t \in [0,1]$. Therefore, the mapping $\mathfrak{P}, \mathfrak{T}$ gratifies all the axioms in Corollary 3. Hence, the mapping $\mathfrak{P}, \mathfrak{T}$ has a unique FP in $\mathfrak{G}(J, \mathbb{R})$. Consequently, the nonlinear I.E. (53) possess a unique common solution.

3 Conclusion

This Article is worried with the reality and uniqueness of CFP for broad rational type $(\vartheta, \psi, \varphi)$ - weak contractive axiom in the framework of b-Metric Space with some example. The pairs of mappings are weakly compatible and one pair is α - admissible w.r.t. the other. Our results broad and improve some well known results in the literature. As an application of our results, we have discussed the existence of common solution of integral equations.

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