Stability of Orthogonally Additive-Quadratic Functional Equation in Multi-Banach Spaces

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Abstract

In this paper, we establish the Hyers-Ulam stability of the following Orthogonally Additive-Quadratic functional equation in Multi-Banach Spaces.

\[ \zeta(2i + j) - \zeta(i + 2j) - \zeta(i + j) - \zeta(j - i) - \zeta(i) + \zeta(j) + \zeta(2j) = 0 \]

with \( i \perp j \) where, \( \perp \) is orthogonality in the sense of Ratz.

1. Introduction

The stability problem of functional equations has a long history. Stability is investigated when one concerns whether a small error of parameters causes a large deviation of the solution. Generally speaking, given a function which satisfies a functional equation approximately called a approximate solution, we ask: Is there a solution of this equation which is close to the approximate solution in some accuracy? An earlier work was done by Hyers [6] in order to answer Ulam’s equation [18] on approximately additive mappings.

During last decades various stability problems for large variety of functional equations have been investigated by several mathematicians. A large list of references concerning in the stability of functional equations can be found, e.g.([1], [2], [6], [7], [8], [10]).


\[ f(x + ny) + f(x - ny) = n^2 f(x + y) + n^2 f(x - y) + 2(1 - n^2)f(x) \]
\[ n^4 - n^2 \frac{12}{12} [f(2y) + f(-2y) - 4f(y) - 4f(-y)] \]

for fixed integers \( n \) with \( n \neq 0, \pm 1 \) in Multi-Banach Spaces.

In 2011, Zhihua Wang, Xiaopei Li and Th. M. Rassias proved the Hyers-Ulam stability of the additive-cubic-quartic functional equations

\[ 11 [f(x + 2y) + f(x - 2y)] = 44 [f(x + y) + f(x - y)] + 12f(3y) - 48f(2y) + 60f(y) - 66f(x) \]

in Multi-Banach Spaces by using fixed point method.

In 2013, Fridoun Moradlou proved the generalized Hyers-Ulam-Rassias stability of the Euler-Lagrange-Jensen Type Additive mapping in Multi-Banach Spaces. In 2015, Xiuzhong Yang, Lidan Chang, Guofen Liu established the orthogonal stability of mixed additive-quadratic jensen type functional equation in Multi-Banach Spaces.

In 2015, Young Ju Jeon and Chang Il Kim investigated the following additive-quadratic functional equation

\[ f(2x + y) - f(x + 2y) - f(x + y) - f(y - x) - f(x) + f(y) + f(2y) = 0 \]

in orthogonality space by using fixed point method.

In 2016, R. Murali, M. Deboral and A. Antony Raj proved the Hyers-Ulam stability of the additive-cubic functional equation

\[ f(2x + y) + f(2x - y) - f(4x) = 2f(x + y) + 2f(x - y) - 8f(2x) + 10f(x) - 2f(-x) \]

for all \( x, y \) with \( x \perp y \) in orthogonal space.

In 2016, Sattar Alizadeh, Fridoun Moradlou proved the generalized Hyers-Ulam-Rassias stability of the quadratic mapping in multi-Banach spaces.

In this paper, we achieve the stability of the orthogonally Additive-Quadratic functional equation

\[ \zeta(2i + j) - \zeta(i + 2j) - \zeta(i + j) - \zeta(j - i) - \zeta(i) + \zeta(j) + \zeta(2j) = 0 \quad (1) \]

with \( i \perp j \) in Multi-Banach Spaces. It is easy to see that the function \( \zeta(i) = ai^2 + bi \) is a solution of (1).

**Theorem 1.1** [3, 14] Let \((\mathcal{X}, d)\) be a complete generalized metric space and let \( \mathcal{F} : \mathcal{X} \rightarrow \mathcal{X} \) be a strictly contractive mapping with Lipschitz constant \( \mathcal{L} < 1 \). Then for each given element \( x \in \mathcal{X} \), either

\[ d(\mathcal{F}^n x, \mathcal{F}^{n+1} x) = \infty \]
for all nonnegative integers \( n \) or there exists a positive integer \( n_0 \) such that

\[
\text{(i)} \quad d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) < \infty \text{ for all } n \geq n_0; \\
\text{(ii)} \quad \text{The sequence } \{\mathcal{J}^n x\} \text{ is convergent to a fixed point } y^* \text{ of } \mathcal{J}; \\
\text{(iii)} \quad y^* \text{ is the unique fixed point of } T \text{ in the set } Y = \{y \in X : d(\mathcal{J}^{n_0} x, y) < \infty\}; \\
\text{(iv)} \quad d(y, y^*) \leq \frac{1}{1-\rho} d(y, \mathcal{J} y) \text{ for all } y \in Y.
\]

Now, let us recall some concepts concerning Multi-Banach spaces.

Let \((\wp, ||\cdot||)\) be a complex normed space, and let \(k \in \mathbb{N}\). We denote by \(\wp^k\) the linear space \(\wp \oplus \wp \oplus \cdots \oplus \wp\) consisting of \(k\)-tuples \((x_1, \ldots, x_k)\) where \(x_1, \ldots, x_k \in \wp\). The linear operations on \(\wp^k\) are defined coordinate wise. The zero element of either \(\wp\) or \(\wp^k\) is denoted by \(0\). We denote by \(\mathbb{N}_k\) the set \(\{1, 2, \ldots, k\}\) and by \(\Psi_k\) the group of permutations on \(k\) symbols.

**Definition 1.2** [4] A Multi-norm on \(\{\wp^k : k \in \mathbb{N}\}\) is a sequence \((\|\cdot\|_k : k \in \mathbb{N})\) such that \(\|\cdot\|_k\) is a norm on \(\wp^k\) for each \(k \in \mathbb{N}\), \(\|x\|_1 = \|x\|\) for each \(x \in \wp\), and the following axioms are satisfied for each \(k \in \mathbb{N}\) with \(k \geq 2\):

1. \(\| (x_{\sigma(1)}, \ldots, x_{\sigma(k)}) \|_k = \| (x_1 \ldots x_k) \|_k\), for \(\sigma \in \Psi_k, x_1, \ldots, x_k \in \wp;\)
2. \(\| (\alpha_1 x_1, \ldots, \alpha_k x_k) \|_k \leq (\max_{i \in \mathbb{N}_k} |\alpha_i|) \| (x_1 \ldots x_k) \|_k\)
   for \(\alpha_1 \ldots \alpha_k \in \mathbb{C}, x_1, \ldots, x_k \in \wp;\)
3. \(\| (x_1, \ldots, x_{k-1}, 0) \|_k = \| (x_1, \ldots, x_{k-1}) \|_{k-1}\), for \(x_1, \ldots, x_{k-1} \in \wp;\)
4. \(\| (x_1, \ldots, x_{k-1}, x_{k-1}) \|_k = \| (x_1, \ldots, x_{k-1}) \|_{k-1}\) for \(x_1, \ldots, x_{k-1} \in \wp.\)

In this case, we say that \((\wp^k, \|\cdot\|_k) : k \in \mathbb{N}\) is a multi-normed space.

Suppose that \((\wp^k, \|\cdot\|_k) : k \in \mathbb{N}\) is a multi-normed spaces, and take \(k \in \mathbb{N}\). We need the following two properties of multi-norms. They can be found in [4].

(a) \(\| (x, \ldots, x) \|_k = \|x\|\), for \(x \in \wp,\)

(b) \(\max_{i \in \mathbb{N}_k} \|x_i\| \leq \sum_{i=1}^k \|x_i\| \leq \sum_{i=1}^k \|x_i\|, \text{ for } x_1, \ldots, x_k \in \wp.\)

It follows from (b) that if \((\wp, ||\cdot||)\) is a Banach space, then \((\wp^k, \|\cdot\|_k)\) is a Banach space for each \(k \in \mathbb{N}\); In this case, \((\wp^k, \|\cdot\|_k) : k \in \mathbb{N}\) is a multi-Banach space.

**Lemma 1.3** [4] Suppose that \(k \in \mathbb{N}\) and \((x_1 \ldots x_k) \in \wp^k\). For each \(j \in \{1 \ldots k\}\), let \((x^j_n)_{n=1}^{\ldots} \) be a sequence in \(\wp\) such that \(\lim_{n \to \infty} x^j_n = x_j\). Then

\[
\lim_{n \to \infty} (x^1_n - y_1, \ldots, x^k_n - y_k) = (x_1 - y_1 \ldots x_k - y_k) \tag{2}
\]

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holds for all \((y_1, ..., y_k) \in \mathbb{V}^k\).

**Definition 1.4** [4] Let \((\mathbb{V}^k, \| \cdot \|_k) : k \in \mathbb{N}\) be a multi-normed space. A sequence \((x_n)\) in \(\mathbb{V}\) is a multi-null sequence if for each \(\eta > 0\), there exists \(n_0 \in \mathbb{N}\) such that

\[
\sup_{k \in \mathbb{N}} \|(x_n, ..., x_{n+k-1})\|_k \leq \eta \quad (n \geq n_0).
\]

Let \(x \in \mathbb{V}\), we say that the sequence \((x_n)\) is multi-convergent to \(x\) in \(\mathbb{V}\) and write \(\lim_{n \to \infty} x_n = x\) if \((x_n - x)\) is a multi-null sequence.

There are several orthogonality notations on a real normed space available. But here, we present the orthogonal concept introduced by Ratz [13]. This is given in the following definition.

**Definition 1.5** Suppose that \(X\) is a vector space (algebraic module) with \(\dim X \geq 2\), and \(\perp\) is a binary relation on \(X\) with the following properties:

1. Totality of \(\perp\) for zero: \(x \perp 0, 0 \perp x\) for all \(x \in X\);
2. Independence: If \(x, y \in X - \{0\}\) and \(x \perp y\), then \(x\) and \(y\) are linearly independent;
3. Homogeneity: If \(x, y \in X\) and \(x \perp y\), then \(\alpha x \perp \beta y\) for all \(\alpha, \beta \in \mathbb{R}\);
4. Thalesian property: If \(P\) is a 2-dimensional subspace of \(X\), \(x \in P\) and \(\lambda \in \mathbb{R}_+\) (which is the set of non-negative real numbers), then there exists \(y_0 \in P\) such that \(x \perp y_0\) and \(x + y_0 \perp \lambda x - y_0\).

The pair \((X, \perp)\) is called an orthogonality space (resp., module). By an orthogonality normed space (normed module) we mean an orthogonality space (resp., module) having a normed (resp., normed module) structure.

**Definition 1.6** Let \(X\) be a set. A function \(d : X \times X \to [0, \infty]\) is called a generalized metric on \(X\) if and only if \(d\) satisfies

- \(d(x, y) = 0\) if and only if \(x = y\);
- \(d(x, y) = d(y, x)\) for all \(x, y \in X\);
- \(d(x, z) \leq d(x, y) + d(y, z)\) for all \(x, y, z \in X\).

**Theorem 1.7** Let \(S\) be an orthogonality space and let \((T^k, \| \cdot \|) : K \in \mathbb{N}\) be a multi-Banach space. Suppose that \(\eta\) is a nonnegative real number and \(\zeta : S \to T\) is

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a mapping satisfying

\[
\sup_{k \in \mathbb{N}} \| (D\zeta(i_1, j_1), ..., D\zeta(i_k, j_k)) \|_k \leq \eta \quad (4)
\]

\(i_1, ..., i_k, j_1, ..., j_k \in S\) and \(i_x \perp j_x \ (x = 1, 2, ..., k)\) and \(f(0) = 0\). Then there exists a unique Orthogonally Additive mapping \(A : S \to T\) such that

\[
\sup_{k \in \mathbb{N}} \| (\zeta(i_1) - A(i_1), ..., \zeta(i_k) - A(i_k)) \|_k \leq \eta \quad (5)
\]

\(i_1, i_2, ..., i_k \in S\).

Proof: Let \(\Lambda = \{ g : S \to T | g(0) = 0 \}\) and introduce the generalized metric \(d\) defined on \(\Lambda\) by

\[
d(u, v) = \inf \left\{ \lambda \in [0, \infty] | \sup_{j \in S} \| u(j_1) - v(j_1), u(j_k) - v(j_k) \|_k \leq \lambda \right\}
\]

Then it is easy to show that \((\Lambda, d)\) is a generalized complete metric space \([11]\).

We define an operator \(J : \Lambda \to \Lambda\) by

\[
J(u)(j) = \frac{1}{2} u(2j) \quad j \in S
\]

We assert that \(J\) is a strictly contractive operator. Given \(u, v \in \Lambda\), let \(\lambda \in [0, \infty]\) be an arbitrary constant with \(d(u, v) \leq \lambda\). By the definition

\[
\sup_{k \in \mathbb{N}} \| (u(j_1) - v(j_1), ..., u(j_k) - v(j_k)) \|_k \leq \lambda \quad j_1, ..., j_k \in S.
\]

Therefore,

\[
\sup_{k \in \mathbb{N}} \| (J u(j_1) - J v(j_1), ..., J u(j_k) - J v(j_k)) \|_k \\
\leq \sup_{k \in \mathbb{N}} \left\| \left( \frac{1}{2} u(2j_1) - \frac{1}{2} v(2j_1), ..., \frac{1}{2} u(2j_k) - \frac{1}{2} v(2j_k) \right) \right\|_k \\
\leq \frac{1}{2} \lambda
\]
Let \( j_1, \ldots, j_k \in S \). Hence, it holds that
\[
d(\mathcal{J}u, \mathcal{J}v) \leq \frac{1}{2} \lambda d(\mathcal{J}u, \mathcal{J}v) \leq \frac{1}{2} d(u, v)
\]
\[\forall u, v \in \Lambda.\]

Letting \( j_1 = j_2 = \ldots = j_k = 0 \) in (4), we obtain that
\[
\sup_{k \in \mathbb{N}} \| (\zeta(2j_1) - 2\zeta(j_1), \ldots, \zeta(2j_k) - 2\zeta(2j_k)) \|_k \leq \eta
\]
(6)

for all \( i_x \in S, i_x \perp 0 \ (x = 1, 2, \ldots, k). \)

Dividing on both sides 2 by (6), we can get
\[
\sup_{k \in \mathbb{N}} \left\| \left( \zeta(j_1) - \frac{1}{2} \zeta(2j_1), \ldots, \zeta(j_k) - \frac{1}{2} \zeta(2j_k) \right) \right\|_k \leq \frac{1}{2} \eta
\]
(7)

This means that \( \mathcal{J} \) is strictly contractive operator on \( \Lambda \) with the Lipschitz constant \( \mathcal{L} = \frac{1}{2} \).

By (7), we have \( d(\mathcal{J} \zeta, \zeta) \leq \frac{1}{2} \eta < \infty \). According to Theorem (1.1), we deduce the existence of a fixed point of \( \mathcal{J} \) that is the existence of mapping \( A : S \to T \) such that
\[
A(2j) = 2A(j) \quad \forall j \in S.
\]

Moreover, we have \( d(\mathcal{J}^n \zeta, A) \to 0 \), which implies
\[
A(q) = \lim_{n \to \infty} \mathcal{J}^n \zeta(j) = \lim_{n \to \infty} \frac{\zeta(2^n j)}{2^n}
\]
for all \( q \in S. \)

Also, \( d(\zeta, A) \leq \frac{1}{1 - \mathcal{L}} d(\mathcal{J} \zeta, \zeta) \) implies the inequality
\[
d(\zeta, A) \leq \frac{1}{1 - \frac{1}{2}} d(\mathcal{J} \zeta, \zeta)
\]
\[
\leq \eta.
\]

Considering Definition (1.5), we have \( 2^n i \perp 2^n j \). Set
\[
i_1 = \ldots = i_k = 2^n i, j_1 = \ldots = j_k = 2^n j
\]
in (4) and divide both sides by $2^n$. Then, using property (a) of multi-norms, we obtain

$$
\|DA(i,j)\| = \lim_{n \to \infty} \frac{1}{2^n} \|D\zeta(2^n i, 2^n j)\|
$$

$$
\leq \lim_{n \to \infty} \frac{\eta}{2^n} = 0
$$

for all $i, j \in S$. Hence $A$ is Additive.

The uniqueness of $A$ follows from the fact that $A$ is the unique fixed point of $J$ with the property that there exists $\ell \in (0, \infty)$ such that

$$
\sup_{k \in \mathbb{N}} \|(\zeta(i_1) - A(i_1), ..., \zeta(i_k) - A(i_k))\|_k \leq \ell
$$

for all $i_1, ..., i_k \in S$. This completes the proof of the Theorem.

**Theorem 1.8** Let $S$ be an orthogonality space and let $(\langle T^k, \| . \| : K \in \mathbb{N} \rangle)$ be a multi-Banach space. Suppose that $\eta$ is a nonnegative real number and $\zeta : S \to T$ is a mapping satisfying the inequality (4). Then there exists a unique Orthogonally Quadratic mapping $Q : S \to T$ such that

$$
\sup_{k \in \mathbb{N}} \|(\zeta(i_1) - Q(i_1), ..., \zeta(i_k) - Q(i_k))\|_k \leq \frac{1}{3} \eta
$$

(8)

$i_1, i_2, ..., i_k \in S$.

Proof: By (6), we obtain

$$
\|(\zeta(2i_1) - 4\zeta(i_1), ..., \zeta(2i_k) - 4\zeta(i_k))\|_k \leq \eta
$$

(9)

Dividing on both side 4 by (9), we can get

$$
\left\| \left( \zeta(i_1) - \frac{1}{4} \zeta(2i_1), ..., \zeta(i_k) - \frac{1}{4} \zeta(2i_k) \right) \right\|_k \leq \frac{1}{4} \eta
$$

(10)

By (10), we have we have $d(J\zeta, \zeta) \leq \frac{1}{4} \eta < \infty$.

Also, $d(Q, \zeta) \leq \frac{1}{1 - L} d(J\zeta, \zeta)$ implies the inequality

$$
d(Q, \zeta) \leq \frac{1}{1 - \frac{3}{4}} d(J\zeta, \zeta) \leq \frac{1}{3} \eta.
$$

The rest of the proof is similar to that of Theorem (1.7).
**Theorem 1.9** Let $S$ be an an orthogonality space and let $((T^k, \|\|) : K \in \mathbb{N})$ be a multi-Banach space. Suppose that $\eta \geq 0$ and $\zeta : S \to T$ is an mapping satisfying
\[
\sup_{k \in \mathbb{N}} \|D\zeta(i_1, j_1) , ..., D\zeta(i_k, j_k)\|_k \leq \eta \tag{11}
\]
for all $i_1, ..., i_k, j_1, ..., j_k \in S$. Then there exist a unique orthogonally additive mapping $A : S \to T$ and a unique orthogonally quadratic mapping $Q : S \to T$ such that
\[
\sup_{k \in \mathbb{N}} \|\zeta(i_1) - A(i_1) - Q(i_1), ..., \zeta(i_k) - A(i_k) - Q(i_k)\|_k \leq \frac{4}{3} \eta \tag{12}
\]
for all $i_1, i_2, ..., i_k \in S$.

Proof: By Theorem (1.7), (1.8) there exist a unique additive mapping $A_0 : S \to T$ and a unique quadratic mapping $Q_0 : S \to T$ such that
\[
\sup_{k \in \mathbb{N}} \|\zeta(i_1) - A_0(i_1), ..., \zeta(i_k) - A_0(i_k)\|_k \leq \eta \quad \quad (13)
\]
and
\[
\sup_{k \in \mathbb{N}} \|\zeta(i_1) - Q_0(i_1), ..., \zeta(i_k) - Q_0(i_k)\|_k \leq \frac{1}{3} \eta \quad \quad (14)
\]
for all $i_1, ..., i_k \in S$. Now from (13) and (14), we get
\[
\sup_{k \in \mathbb{N}} \|\zeta(i_1) + A_0(i_1) - Q_0(i_1), ..., \zeta(i_k) + A_0(i_k) - Q_0(i_k)\|_k \leq \frac{4}{3} \eta \quad \quad (15)
\]
for all $i_1, ..., i_k \in S$. Thus we obtain (12) by defining $A(i) = -A_0(i)$ and $Q(i) = Q_0(i)$. The uniqueness of $A$ and $Q$ is easy to show.

**References**


