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Super Metric Spaces and Fixed Point Results using $(E.A)$ and $(CLR_{\mathscr{P}})$ Properties

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Abstract

In this article, we established the common fixed point results by using property $(E.A)$ and $(CLR_{\mathscr{P}})$ property in the framework of Super metric spaces. The obtained results extend several well-defined findings in classical metric spaces to the more general setting of Super metric spaces. Furthermore, suitable example is provided to demonstrate the effectiveness and applicability of the proposed results.

Key words: Common fixed point, super metric spaces, weakly compatible maps, property $(E.A)$, $(CLR_{\mathscr{P}})$ property.

AMS classification: 47H10 , 54H25

1 Introduction

Fixed point$(\mathscr{F}\mathscr{P})$ theory has attracted prominent focus from researchers because of its wide range of applications in multiple areas of science. $\mathscr{F}\mathscr{P}$ theory techniques help researchers to analyze non-linear problems, optimization algorithms, dynamical systems and neural network models. Banach Contraction Principle is regarded as one of essential results in $\mathscr{F}\mathscr{P}$ theory and has played a major role in the progress of nonlinear analysis. This principle promise the existence together with uniqueness of $\mathscr{F}\mathscr{P}$ for contraction mappings defined on complete metric spaces. Owing to its significance, many researchers have generalized and extended this classical result by establishing different types of contraction conditions and generalized metric structures to address more complex nonlinear problems.

Recently, In 2022, E. Karapinar and F. Khojasteh[10] introduced the concept of Super metric spaces $(\mathscr{S}\mathscr{M}\mathscr{S})$ as an extension of JS-metric spaces. $\mathscr{S}\mathscr{M}\mathscr{S}$ offer a more flexible structure that allows experts to tackle problems that traditional methods

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cannot, such as those related to non-linear dynamics or multiscale complexity. Further, E. Karapinar and A. Fulga [9] investigate contractions in rational form in the framework of $\mathscr{SMS}$. N. Hooda et al.[6] generalized the results of E. Karapinar and F. Khojasteh[10] and E. Karapinar and A. Fulga[9] in $\mathscr{SMS}$ by using the control functions and weakly compatible (**WC**) mappings. In 2024, K. Abodayeh et al.[3] explore Ćirić-type generalized F-contractions, almost F-contraction and combination of these contractions in the structure of $\mathscr{SMS}$.

2 Main Results

The following definitions are used in the subsequent sections.

Definition 2.1 [10] Consider a nonempty set $\mathscr{S}$ and a mapping $m : \mathscr{S} \times \mathscr{S} \to [0, \infty)$ is known as a Super metric if it fulfills the following properties:

(m_1) $m(\kappa, \nu) = 0 \implies \kappa = \nu$, $\forall\ \kappa, \nu \in \mathscr{S}$;
(m_2) $m(\kappa, \nu) = m(\nu, \kappa)$, $\forall\ \kappa, \nu \in \mathscr{S}$;
(m_3) $\exists\ s \geq 1$ s.t. $\forall\ \nu \in \mathscr{S}$ $\exists$ distinct sequences $\{\kappa_n\}$, $\{\nu_n\} \subset \mathscr{S}$ with $m(\kappa_n, \nu_n) \to 0$ when n tends to infinity s.t.

$$\lim_{n\to\infty} \sup\, m(\nu_n, \nu) \leq s \lim_{n\to\infty} \sup\, m(\kappa_n, \nu).$$

Then, we call $(\mathscr{S}, m)$ a $\mathscr{SMS}$.

Definition 2.2 [10] Let $(\mathscr{S}, m)$ be a $\mathscr{SMS}$ and let $\{\kappa_n\}$ be a sequence in $\mathscr{S}$. We say

(i) $\{\kappa_n\}$ converges to κ in $\mathscr{S} \iff m(\kappa_n, \kappa) \to 0$ as $n \to \infty$.
(ii) $\{\kappa_n\}$ is called a Cauchy sequence in $\mathscr{S} \iff \lim_{n\to\infty} sup\{m(\kappa_n, \kappa_m); m > n\} = 0$.
(iii) $(\mathscr{S}, m)$ is said to be complete $\mathscr{SMS} \iff$ every cauchy sequence in $\mathscr{S}$ converges to a point in $\mathscr{S}$.

Proposition 2.3 [10] let $(\mathscr{S}, m)$ be a complete $\mathscr{SMS}$ and let $\mathscr{P} : \mathscr{S} \to \mathscr{S}$ be a mapping. Suppose that $0 < k < 1$ s.t.

$$m(\mathscr{P}\kappa, \mathscr{P}\nu) \leq km(\kappa, \nu) \quad \forall \quad \kappa, \nu \in \mathscr{S}.$$

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Then $\mathscr{P}$ possesses a unique $\mathscr{F}\mathscr{P}$ in $\mathscr{S}$.

Proposition 2.4 [9] On a $\mathscr{S}\mathscr{M}\mathscr{S}$, the limit of a convergent sequence is unique.

Definition 2.5 [8] A pair $(\mathscr{Q},\mathscr{P})$ of self mappings of metric space $(\mathscr{S},m)$ is said to be $\boldsymbol{WC}$ if $\mathscr{Q}\kappa = \mathscr{P}\kappa$ for some $\kappa \in \mathscr{S} \implies \mathscr{Q}\mathscr{P}\kappa = \mathscr{P}\mathscr{Q}\kappa$.

Definition 2.6 [2] Let $\mathscr{Q}$ and $\mathscr{P}$ be self mappings on $\mathscr{S}$, if $w = \mathscr{Q}\kappa = \mathscr{P}\kappa$ for some $\kappa \in \mathscr{S}$, then κ is known as a coincidence point ($\mathscr{C}\mathscr{P}$) of $\mathscr{P}$ and $\mathscr{Q}$ and w is referred to as the point of coincidence of the mappings $\mathscr{Q}$ and $\mathscr{P}$.

Proposition 2.7 [2] Let $\mathscr{Q}$ and $\mathscr{P}$ be $\boldsymbol{WC}$ self mappings on $\mathscr{S}$, if $\mathscr{Q}$ and $\mathscr{P}$ share a unique point of coincidence w, then w is also the unique common fixed point ($\mathscr{C}\mathscr{F}\mathscr{P}$) of $\mathscr{Q}$ and $\mathscr{P}$.

Definition 2.8 [1] Let $\mathscr{Q}$ and $\mathscr{P}$ be self mappings on $\mathscr{S}$. The pair $(\mathscr{Q},\mathscr{P})$ satisfies the property (prop.) $(E.A)$ if $\exists$ a sequence $\{\kappa_n\}$ in $\mathscr{S}$ s.t.

$$\lim_{n\to\infty} \mathscr{Q}\kappa_n = \lim_{n\to\infty} \mathscr{P}\kappa_n = t \text{ for some } t \in \mathscr{S}.$$

Definition 2.9 [4] Let $\mathscr{Q}$ and $\mathscr{P}$ be self mappings on $\mathscr{S}$. The pair $(\mathscr{Q},\mathscr{P})$ satisfies the $(CLR_{\mathscr{P}})$ prop. if $\exists$ a sequence $\{\kappa_n\}$ in $\mathscr{S}$ s.t.

$$\lim_{n\to\infty} \mathscr{Q}\kappa_n = \lim_{n\to\infty} \mathscr{P}\kappa_n = \mathscr{P}\kappa \text{ for some } \kappa \in \mathscr{S}.$$

Proposition 2.10 [7] Let $(\mathscr{S},m)$ be a complete metric space. Let $\mathscr{P}$ be a continuous mapping on $\mathscr{S}$ and $\mathscr{Q}$ be any mapping on $\mathscr{S}$ that commutes with $\mathscr{P}$. Further let $\mathscr{P}$ and $\mathscr{Q}$ satisfy $\mathscr{Q}(\mathscr{S}) \subset \mathscr{P}(\mathscr{S})$ and $\exists$ a constant $\beta \in (0,1)$ s.t. for every κ, $\nu \in \mathscr{S}$,

$$m(\mathscr{Q}\kappa, \mathscr{Q}\nu) \le \beta m(\mathscr{P}\kappa, \mathscr{P}\nu).$$

Then, $\mathscr{P}$ and $\mathscr{Q}$ admit a unique $\mathscr{C}\mathscr{F}\mathscr{P}$.

Proposition 2.11 [5] Let $(\mathscr{S},m)$ be a metric space that is complete. Let $\mathscr{P}$ be a continuous mapping on $\mathscr{S}$ and $\mathscr{Q}$ be any mapping on $\mathscr{S}$ that commutes with $\mathscr{P}$.

Further let $\mathscr{P}$ and $\mathscr{Q}$ satisfy $\mathscr{Q}(\mathscr{S}) \subset \mathscr{P}(\mathscr{S})$ and $\exists$ a constant $\beta \in (0,1)$ s.t. for every κ, $\nu \in \mathscr{S}$,

$$m(\mathscr{Q}\kappa, \mathscr{Q}\nu) \leq \beta \mathscr{M}(\kappa, \nu),$$

where

$$\mathscr{M}(\kappa, \nu) = max\{m(\mathscr{P}\kappa, \mathscr{P}\nu), m(\mathscr{P}\kappa, \mathscr{Q}\kappa), m(\mathscr{P}\kappa, \mathscr{Q}\nu), m(\mathscr{P}\nu, \mathscr{Q}\nu), m(\mathscr{P}\nu, \mathscr{Q}\kappa)\}.$$

Then $\mathscr{P}$ and $\mathscr{Q}$ admit a unique $\mathscr{C}\mathscr{F}\mathscr{P}$.

Our main aim is to extend and generalize the $\mathscr{F}\mathscr{P}$ results of [5] and [7] in the framework of $\mathscr{S}\mathscr{M}\mathscr{S}$.

Theorem 2.12 Let $\mathscr{Q}$ and $\mathscr{P}$ be self mappings defined on the $\mathscr{S}\mathscr{M}\mathscr{S}$ $(\mathscr{S}, m)$, fulfills the axioms given below:

(i) $m(\mathscr{Q}\kappa, \mathscr{Q}\nu) \leq k$ m$(\mathscr{P}\kappa, \mathscr{P}\nu)$ $\forall$ κ, $\nu \in \mathscr{S}$ and $k \in (0,1)$
(ii) $\mathscr{Q}$ and $\mathscr{P}$ possess the prop. $(E.A)$
(iii) $\mathscr{P}(\mathscr{S})$ is closed in $\mathscr{S}$.

Then $\mathscr{Q}$ and $\mathscr{P}$ admit a unique $\mathscr{C}\mathscr{F}\mathscr{P}$ in $\mathscr{S}$, provided $\mathscr{Q}$ and $\mathscr{P}$ are ***WC*** mappings.
Proof: As $\mathscr{Q}$ and $\mathscr{P}$ satisfy prop. $(E.A)$, $\exists$ a sequence $\{\kappa_n\} \in \mathscr{S}$ s.t.

$$\lim_{n\to\infty} \mathscr{Q}\kappa_n = \lim_{n\to\infty} \mathscr{P}\kappa_n = t \text{ for some } t \in \mathscr{S}.$$

Since $\mathscr{P}(\mathscr{S})$ is closed in $\mathscr{S}$, therefore $\exists$ $u \in \mathscr{P}(\mathscr{S})$ s.t. $\lim_{n\to\infty} \mathscr{P}(\kappa_n) = \mathscr{P}u = t = \lim_{n\to\infty} \mathscr{Q}(\kappa_n)$.
We claim that $\mathscr{Q}u = \mathscr{P}u = t$
Now, applying the contractive condition, $m(\mathscr{Q}\kappa_n, \mathscr{Q}u) \leq k\ m(\mathscr{P}\kappa_n, \mathscr{P}u)$
By using (m_3), we have, $m(t, \mathscr{Q}u) \leq 0 \implies \mathscr{Q}u = \mathscr{P}u = t$.
So, $\mathscr{Q}\mathscr{P}u = \mathscr{P}\mathscr{Q}u$ as $\mathscr{Q}$ and $\mathscr{P}$ are ***WC***.
Because $\mathscr{Q}\mathscr{P}u = \mathscr{P}\mathscr{Q}u$, it follows that

$$m(\mathscr{Q}u, \mathscr{Q}\mathscr{Q}u) \leq km(\mathscr{P}u, \mathscr{P}\mathscr{Q}u) = km(\mathscr{Q}u, \mathscr{Q}\mathscr{P}u) = km(\mathscr{Q}u, \mathscr{Q}\mathscr{Q}u)$$

Consequently, $m(\mathscr{Q}u, \mathscr{Q}\mathscr{Q}u)(1-k) \leq 0$. Since $k \in (0,1)$, $\mathscr{Q}u = \mathscr{Q}\mathscr{Q}u = \mathscr{P}\mathscr{Q}u$, i.e. $\mathscr{Q}u$ is $\mathscr{C}\mathscr{F}\mathscr{P}$ of $\mathscr{Q}$ and $\mathscr{P}$.

Uniqueness: Suppose that κ and ν are two $\mathscr{CFP}$ of $\mathscr{Q}$ and $\mathscr{P}$ i.e. $\mathscr{P}\kappa = \mathscr{Q}\kappa = \kappa$ and $\mathscr{P}\nu = \mathscr{Q}\nu = \nu$.
Now, applying the contractive condition, $m(\kappa,\nu) = m(\mathscr{Q}\kappa,\mathscr{Q}\nu) \leq km(\mathscr{P}\kappa,\mathscr{P}\nu) = k(\kappa,\nu) \implies \kappa = \nu$.

Theorem 2.13 Let $\mathscr{Q}$ and $\mathscr{P}$ be self mappings defined on the $\mathscr{SMS}$ $(\mathscr{S},m)$, fulfills the axioms given below:

(i)
$$m(\mathscr{Q}\kappa,\mathscr{Q}\nu) \leq k\max\Big\{m(\mathscr{P}\kappa,\mathscr{P}\nu), m(\mathscr{P}\kappa,\mathscr{Q}\kappa), m(\mathscr{P}\kappa,\mathscr{Q}\nu),$$
$$m(\mathscr{P}\nu,\mathscr{Q}\kappa), m(\mathscr{P}\nu,\mathscr{Q}\nu)\Big\},$$

$\forall\ \kappa,\nu \in \mathscr{S}$ and $k \in (0,\frac{1}{2})$,

(ii) $\mathscr{Q}$ and $\mathscr{P}$ possess the prop. $(E.A)$,

(iii) $\mathscr{P}(\mathscr{S})$ is closed in $\mathscr{S}$.

Then $\mathscr{Q}$ and $\mathscr{P}$ admit a unique $\mathscr{CFP}$ in $\mathscr{S}$, provided $\mathscr{Q}$ and $\mathscr{P}$ are ***WC*** mappings.
Proof: As $\mathscr{Q}$ and $\mathscr{P}$ satisfy prop. $(E.A)$, $\exists$ a sequence $\{\kappa_n\} \in \mathscr{S}$ s.t.

$$\lim_{n\to\infty} \mathscr{Q}\kappa_n = \lim_{n\to\infty} \mathscr{P}\kappa_n = t \text{ for some } t \in \mathscr{S}.$$

Since $\mathscr{P}(\mathscr{S})$ is closed in $\mathscr{S}$, therefore $\exists\ \ u \in \mathscr{P}(\mathscr{S})$ s.t.
$\lim_{n\to\infty} \mathscr{P}(\kappa_n) = \mathscr{P}u = t = \lim_{n\to\infty} \mathscr{Q}(\kappa_n)$.
We claim that $\mathscr{Q}u = \mathscr{P}u = t$
Now, applying the contractive condition,

$$m(\mathscr{Q}\kappa_n,\mathscr{Q}u) \leq k\max\Big\{m(\mathscr{P}\kappa_n,\mathscr{P}u), m(\mathscr{P}\kappa_n,\mathscr{Q}\kappa_n),$$
$$m(\mathscr{P}\kappa_n,\mathscr{Q}u), m(\mathscr{P}u,\mathscr{Q}\kappa_n), m(\mathscr{P}u,\mathscr{Q}u)\Big\}$$

Taking limit n $\to \infty$ and using the postulates of $\mathscr{SMS}$ together with convergence properties, we obtain, $m(t,\mathscr{Q}u) \leq km(t,\mathscr{Q}u) \implies \mathscr{Q}u = \mathscr{P}u = t$.
So, $\mathscr{QP}u = \mathscr{PQ}u$ as $\mathscr{Q}$ and $\mathscr{P}$ are ***WC***.

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Because $\mathscr{Q}\mathscr{P}u = \mathscr{P}\mathscr{Q}u$, it follows that

$$\begin{aligned} m(\mathscr{Q}u, \mathscr{Q}\mathscr{Q}u) &\leq k\max\Big\{m(\mathscr{P}u, \mathscr{P}\mathscr{Q}u), m(\mathscr{P}u, \mathscr{Q}u), \\ &\qquad m(\mathscr{P}u, \mathscr{Q}\mathscr{Q}u), m(\mathscr{P}Su, \mathscr{Q}u), m(\mathscr{Q}u, \mathscr{Q}\mathscr{Q}u)\}\Big\} \\ &= km(\mathscr{Q}u, \mathscr{Q}\mathscr{P}u) \\ &= km(\mathscr{Q}u, \mathscr{Q}\mathscr{Q}u). \end{aligned}$$

Consequently, $m(\mathscr{Q}u, \mathscr{Q}\mathbf{Q}u)(1-k) \leq 0$. Since $k \in (0, \frac{1}{2})$, $\mathscr{Q}u = \mathscr{Q}\mathscr{Q}u = \mathscr{P}\mathscr{Q}u$, i.e. $\mathscr{Q}u$ is $\mathscr{C}\mathscr{F}\mathscr{P}$ of $\mathscr{Q}$ and $\mathscr{P}$.

Uniqueness: Suppose that κ and ν are two $\mathscr{C}\mathscr{F}\mathscr{P}$ of $\mathscr{Q}$ and $\mathscr{P}$ i.e. $\mathscr{P}\kappa = \mathscr{Q}\kappa = \kappa$ and $\mathscr{P}\nu = \mathscr{Q}\nu = \nu$.

Now, applying the contractive condition,

$$\begin{aligned} m(\kappa, \nu) &= m(\mathscr{Q}\kappa, \mathscr{Q}\nu) \\ &\leq k\max\Big\{m(\mathscr{P}\kappa, \mathscr{P}\nu), m(\mathscr{P}\kappa, \mathscr{Q}\kappa), m(\mathscr{P}\kappa, \mathscr{Q}\nu), \\ &\qquad m(\mathscr{P}\nu, \mathscr{Q}\kappa), m(\mathscr{P}\nu, \mathscr{Q}\nu)\}\Big\} \\ &= km(\kappa, \nu) \\ &\implies \kappa = \nu. \end{aligned}$$

Remark 2.14 It is important to observe that the mappings, obeying the prop. $(E.A)$ need not obey the range inclusion condition of one mapping into another, which is generally assumed for ***WC*** maps. Furthermore, continuity assumption is not essential for establishing the range inclusion condition, thereby reducing the need for commutativity at $\mathscr{C}\mathscr{P}$s. In addition, the completeness requirement on the underlying space can be weakened to the closedness of the range.

Theorem 2.15 Let $\mathscr{Q}$ and $\mathscr{P}$ be self mappings defined on the $\mathscr{S}\mathscr{M}\mathscr{S}$ $(\mathscr{S}, m)$ fulfills the axioms given below:

(i) $m(\mathscr{Q}\kappa, \mathscr{Q}\nu) \leq k$ m$(\mathscr{P}\kappa, \mathscr{P}\nu)$ $\forall$ $\kappa, \nu \in \mathscr{S}$ and $k \in (0, 1)$

(ii) $(CLR_{\mathscr{P}})$ prop.

Then $\mathscr{Q}$ and $\mathscr{P}$ admit a unique $\mathscr{C}\mathscr{F}\mathscr{P}$ in $\mathscr{S}$, provided $\mathscr{Q}$ and $\mathscr{P}$ are ***WC***.

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Proof: Since $\mathscr{Q}$ and $\mathscr{P}$ satisfy $(CLR_{\mathscr{P}})$ prop., $\exists$ a sequence $\{\kappa_n\}$ in $\mathscr{S}$ s.t.

$$\lim_{n\to\infty} \mathscr{Q}\kappa_n = \lim_{n\to\infty} \mathscr{P}\kappa_n = \mathscr{P}u = t \text{ for some } u, t \in \mathscr{S}.$$

Now, applying the contractive condition, $m(\mathscr{Q}\kappa_n, \mathscr{Q}u) \leq k\ m(\mathscr{P}\kappa_n, \mathscr{P}u)$
By using (m_3), we have, $m(t, \mathscr{Q}u) \leq 0 \implies \mathscr{Q}u = \mathscr{P}u = t$.
So, $\mathscr{Q}\mathscr{P}u = \mathscr{P}\mathscr{Q}u$ as $\mathscr{Q}$ and $\mathscr{P}$ are $\boldsymbol{WC}$.
Because $\mathscr{Q}\mathscr{P}u = \mathscr{P}\mathscr{Q}u$, it follows that

$$m(\mathscr{Q}u, \mathscr{Q}\mathscr{Q}u) \leq km(\mathscr{P}u, \mathscr{P}\mathscr{Q}u) = km(\mathscr{Q}u, \mathscr{Q}\mathscr{P}u) = km(\mathscr{Q}u, \mathscr{Q}\mathscr{Q}u)$$

Consequently, $m(\mathscr{Q}u, \mathscr{Q}\mathscr{Q}u)(1-k) \leq 0$. Since $k \in (0,1)$, $\mathscr{Q}u = \mathscr{Q}\mathscr{Q}u = \mathscr{P}\mathscr{Q}u$, i.e. $\mathscr{Q}u$ is $\mathscr{C}\mathscr{F}\mathscr{P}$ of $\mathscr{Q}$ and $\mathscr{P}$.
Uniqueness: Suppose that κ and ν are two $\mathscr{C}\mathscr{F}\mathscr{P}$ of $\mathscr{Q}$ and $\mathscr{P}$ i.e. $\mathscr{P}\kappa = \mathscr{Q}\kappa = \kappa$ and $\mathscr{P}\nu = \mathscr{Q}\nu = \nu$.
Now, applying the contractive condition,

$$\begin{aligned} m(\kappa, \nu) &= m(\mathscr{Q}\kappa, \mathscr{Q}\nu) \\ &\leq km(\mathscr{P}\kappa, \mathscr{P}\nu) \\ &= k(\kappa, \nu) \\ &\implies \kappa = \nu. \end{aligned}$$

Theorem 2.16 let $\mathscr{Q}$ and $\mathscr{P}$ be self mappings defined on the $\mathscr{S}\mathscr{M}\mathscr{S}$ $(\mathscr{S}, m)$, fulfills the axioms given below:

(i)
$$\begin{aligned} m(\mathscr{Q}\kappa, \mathscr{Q}\nu) \leq k \max\Big\{ & m(\mathscr{P}\kappa, \mathscr{P}\nu), m(\mathscr{P}\kappa, \mathscr{Q}\kappa), m(\mathscr{P}\kappa, \mathscr{Q}\nu), \\ & m(\mathscr{P}\nu, \mathscr{Q}\kappa), m(\mathscr{P}\nu, \mathscr{Q}\nu)\Big\}, \end{aligned}$$

$\forall\ \kappa, \nu \in \mathscr{S}$ and $k \in (0, \frac{1}{2})$,
(ii) $(CLR_{\mathscr{P}})$ prop.

Then $\mathscr{Q}$ and $\mathscr{P}$ admit a unique $\mathscr{C}\mathscr{F}\mathscr{P}$ in $\mathscr{S}$, provided $\mathscr{Q}$ and $\mathscr{P}$ are $\boldsymbol{WC}$ maps.
Proof: Since $\mathscr{Q}$ and $\mathscr{P}$ satisfy $(CLR_{\mathscr{P}})$ prop., $\exists$ a sequence $\{\kappa_n\}$ in $\mathscr{S}$ s.t.

$$\lim_{n\to\infty} \mathscr{Q}\kappa_n = \lim_{n\to\infty} \mathscr{P}\kappa_n = \mathscr{P}u = t \text{ for some } u, t \in \mathscr{S}.$$

Now, applying the contractive condition,

$$m(\mathscr{Q}\kappa_n, \mathscr{Q}u) \leq k\max\Big\{m(\mathscr{P}\kappa_n, \mathscr{P}u), m(\mathscr{P}\kappa_n, \mathscr{Q}\kappa_n), \\ m(\mathscr{P}\kappa_n, \mathscr{Q}u), m(\mathscr{P}u, \mathscr{Q}\kappa_n), m(\mathscr{P}u, \mathscr{Q}u)\Big\}$$

Taking limit n $\to \infty$ and using the postulates of $\mathscr{S}\mathscr{M}\mathscr{S}$ together with convergence properties, we obtain, $m(t, \mathscr{Q}u) \leq km(t, \mathscr{Q}u) \implies \mathscr{Q}u = \mathscr{P}u = t$.
So, $\mathscr{Q}\mathscr{P}u = \mathscr{P}\mathscr{Q}u$ as $\mathscr{Q}$ and $\mathscr{P}$ are $\boldsymbol{WC}$.
Because $\mathscr{Q}\mathscr{P}u = \mathscr{P}\mathscr{Q}u$, it follows that

$$\begin{aligned} m(\mathscr{Q}u, \mathscr{Q}\mathscr{Q}u) &\leq k\max\Big\{m(\mathscr{P}u, \mathscr{P}\mathscr{Q}u), m(\mathscr{P}u, \mathscr{Q}u), \\ &\qquad m(\mathscr{P}u, \mathscr{Q}\mathscr{Q}u), m(\mathscr{P}Su, \mathscr{Q}u), m(\mathscr{Q}u, \mathscr{Q}\mathscr{Q}u)\}\Big\} \\ &= km(\mathscr{Q}u, \mathscr{Q}\mathscr{P}u) \\ &= km(\mathscr{Q}u, \mathscr{Q}\mathscr{Q}u). \end{aligned}$$

Consequently, $m(\mathscr{Q}u, \mathscr{Q}\mathscr{Q}u)(1-k) \leq 0$. Since $k \in (0, \frac{1}{2})$, $\mathscr{Q}u = \mathscr{Q}\mathscr{Q}u = \mathscr{P}\mathscr{Q}u$, i.e. $\mathscr{Q}u$ is $\mathscr{C}\mathscr{F}\mathscr{P}$ of $\mathscr{Q}$ and $\mathscr{P}$.
Uniqueness: Suppose that κ and ν are two $\mathscr{C}\mathscr{F}\mathscr{P}$ of $\mathscr{Q}$ and $\mathscr{P}$ i.e. $\mathscr{P}\kappa = \mathscr{Q}\kappa = \kappa$ and $\mathscr{P}\nu = \mathscr{Q}\nu = \nu$.
Now, applying the contractive condition,

$$\begin{aligned} m(\kappa, \nu) &= m(\mathscr{Q}\kappa, \mathscr{Q}\nu) \\ &\leq k\max\Big\{m(\mathscr{P}\kappa, \mathscr{P}\nu), m(\mathscr{P}\kappa, \mathscr{Q}\kappa), m(\mathscr{P}\kappa, \mathscr{Q}\nu), \\ &\qquad m(\mathscr{P}\nu, \mathscr{Q}\kappa), m(\mathscr{P}\nu, \mathscr{Q}\nu)\}\Big\} \\ &= km(\kappa, \nu) \\ &\implies \kappa = \nu. \end{aligned}$$

Example 2.17 Let $\mathscr{S} = [1, 3]$ and define

$$m(\kappa, \nu) = \begin{cases} \kappa\nu, & \kappa \neq \nu \\ 0, & \kappa = \nu \end{cases}$$

Then, $(\mathscr{S}, m)$ is a $\mathscr{S}\mathscr{M}\mathscr{S}$ as already shown in [9]. Now, consider $\mathscr{P}, \mathscr{Q} : \mathscr{S} \to \mathscr{S}$ as follows

$$\mathscr{Q}(\kappa) = 1 \text{ and } \mathscr{P}(\kappa) = \frac{\kappa+1}{2}$$

Thus for $k = 0.9$, we obtain that $\mathscr{P}$ and $\mathscr{Q}$ satisfy the assumed contractive condition in theorem (2.13) and theorem (2.16).
Also, for the sequence $\{\kappa_n\} = 1 + \frac{1}{n}$, we have

$$\lim_{n\to\infty} \mathscr{Q}\kappa_n = \lim_{n\to\infty} \mathscr{P}\kappa_n = \mathscr{P}(1) = 1$$

i.e. $\mathscr{P}$ and $\mathscr{Q}$ satisfy the prop. $(E.A)$ and $(CLR_{\mathscr{P}})$ prop..
Now, for $\kappa = 1$, $\mathscr{P}\kappa = \mathscr{Q}\kappa$ and $\mathscr{P}\mathscr{Q}\kappa = \mathscr{Q}\mathscr{P}\kappa$. So, 1 serves as the only point of coincidence of $\mathscr{P}$ and $\mathscr{Q}$. As every requirement of theorem (2.13) and theorem (2.16) holds true. Therefore, 1 is the unique $\mathscr{C}\mathscr{F}\mathscr{P}$ of $\mathscr{P}$ and $\mathscr{Q}$.

3 Conclusion

We have generalized and extended the results of [5] and [7] in the setting of $\mathscr{S}\mathscr{M}\mathscr{S}$ using prop. $(E.A)$ and $(CLR_{\mathscr{P}})$ prop.. Relevant example is also given to show the applicability of results of the theorem (2.13) and theorem (2.16).

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