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On the Oscillation of Fractional Order Emden-Fowler q-Difference Equations

N. Nagajothi[1*], V. Sadhasivam[2], P. Mahalakshmi[3]
[1,2,3]PG & Research Department of Mathematics,
Thiruvalluvar Government Arts College,
Rasipuram - 637 401, Namakkal (Dt), Tamil Nadu, India.

Abstract

In this article, we study the oscillatory behavior of fractional order Emden - Fowler q-difference equations of the form

$$D_q\left[r(t)\left({}^cD_q^{\alpha}z(t)\right)\right] + \phi(t)\left|x(\sigma(t))\right|^{\gamma-1}x(\sigma(t)) = 0, \quad t \geq t_0,$$

where $z(t) = x(t) + p(t)x(t-\tau)$, ${}^cD_q^{\alpha}$ denotes the Caputo q-fractional derivative of order α, $0 < \alpha \leq 1$. Using the generalized Riccati technique, new oscillation criteria are established.

Key words: Oscillation, Fractional differential equations, Neutral, q- Calculus.

AMS classification: 34C10, 35A08, 35K11.

1. Introduction

Fractional differential equations can be found in extensive range of many different subject areas. There are different concepts of fractional derivatives such as Riemann - Liouville and Caputo fractional derivatives are widely used. The Caputo fractional derivatives are based on integral expressions and gamma functions which are nonlocal. Fractional theory and its applications are mentioned many papers and monographs, we refer [1, 12, 15, 17, 24, 26, 27, 28, 29, 33].

Quantum calculus received a great attention and most of the published work has been interested in some problems of q-difference equations. The study of q - difference equations have been initiated by Jackson [22]. The paper of Carmichael [14]. See [2, 3, 4, 5, 7, 8, 9, 10, 11, 14, 16, 20, 22, 23, 30] and the references cited therein. The Emden-Fowler equations have been considered one of the important classical objects in the theory of differential equations. This type of equations has variety of

[1*]nagajothi006@gmail.com, [2]ovsadha@gmail.com, [3]mahahtc7409@gmail.com

interesting physical applications occuring in astro physics and atomic physics. See [6, 13, 18, 19, 21, 25, 31, 32, 34, 35] and the references cited therein. Li et al. [25] considered the Emden-Fowler neutral delay differential equation of the form

$$(r(t)(x(t)+p(t)x(t-\tau))')' + q(t)x^{\gamma}(\sigma(t)) = 0.$$

In [6], the researchers investigated the oscillatory behavior of second-order Emden-Fowler neutral delay differential equations of the form

$$((r(t)(x(t)+p(t)x(t-\tau))')^{\alpha})' + q(t)x^{\gamma}(\sigma(t)) = 0.$$

. In this paper, we investigate the following fractional order Emden-Fowler neutral delay q-difference equation

$$D_q\left[r(t)\left({}^{c}D_q^{\alpha}z(t)\right)\right] + \phi(t)\left|x(\sigma(t))\right|^{\gamma-1}x(\sigma(t)) = 0, \quad t \geq t_0, \tag{1}$$

where $z(t) = x(t) + p(t)x(t-\tau), {}^{c}D_q^{\alpha}$ denotes the Caputo q-fractional derivative of order α, $0 < \alpha \leq 1$. We assume the following conditions throughout this paper without mentioning that
(A_1) $\gamma \in \mathbb{R}$, where $\mathbb{R}$ is the set of all ratios of odd positive integers ;
(A_2) $r \in C\left([t_0,\infty);(0,\infty)\right), p, \phi \in C\left([t_0,\infty);\mathbb{R}\right), 0 \leq p(t) < 1, \phi(t) \geq 0$, and q is a not identically zero for large t;
(A_3) $\tau, \sigma \in C([t_0,\infty),\mathbb{R}), \tau(t) \leq t, \sigma(t) \leq t.$

By a solution of (1) we mean a nontrivial function x satisfying (1) for $t \geq t_x \geq t_0$. in the sequel, we assume that solutions of (1) exist and can be continued indefinitely to the right. A solution of (1) is called oscillatory if it has arbitrarily on $[t_x,\infty)$; Otherwise, it is called nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillory.
There is no work done on the oscillation of q-fractional Emden-Fowler equation. Our main aim of this paper is to establish new oscillation criteria for (1) by using generalized Riccati technique method.

2. Preliminaries

We state some definitions and fundamental results on quantum fractional calculus, see [9,23,30] and the references cited therein.

Definition 2.1 Assume $\nu \geq 0, q \in (0,1)$, and a function g is defined on the interval $[0,1]$. The fractional order q-integral of the Riemann-Liouville type is $(I_q^0 g)(t) = g(t)$

[1*]nagajothi006@gmail.com, [2]ovsadha@gmail.com, [3]mahahtc7409@gmail.com

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and $(I_q^\nu g)(t) = \int_0^t \frac{(t-qs)^{(\nu-1)}}{\Gamma_q(\nu)} g(s) d_q s, \quad \nu > 0, \quad t \in [0,1]$, where
$\Gamma_q(\nu) = \frac{(1-q)^{(\nu-1)}}{(1-q)^{\nu-1}}, \quad q \in (0,1)$ and satisfies the relation: $\Gamma_q(\nu+1) = [\nu]_q \Gamma_q(\nu)$, with

$$[\nu]_q = \frac{q^\nu - 1}{q-1}, \quad (1-q)^{(0)} = 1, \quad (1-q)^{(n)} = \Pi_{k=0}^{n-1}(1-q^{k+1}),$$

$n \in \mathbb{N}$. In general, if $\alpha \in \mathbb{R}$, then

$$(1-q)^{(\alpha)} = \Pi_{i=0}^{\infty} \frac{(1-q^{i+1})}{(1-q^{1+\alpha+i})}.$$

Now, we define the q-derivative of a real valued function g as

$$D_q g(t) = \frac{g(t) - g(qt)}{(1-q)t}, \quad t \neq 0, \quad D_q g(0) = \lim_{n\to\infty} \frac{g(sq^n) - g(0)}{sq^n}, s \neq 0, for \quad q \in (0,1).$$

Definition 2.2 The Riemann-Liouville type of the fractional q-derivative of the of order $\nu \geq 0$ is defined by $(D_q^0 g)(t) = g(t)$ and

$$(D_q^\nu g)(t) = (D_q^{[\nu]} I_q^{[\nu]-\nu} g)(t), \quad \nu > 0$$

where $[\nu]$ is the smallest integer greater than or equal to ν.

Definition 2.3 The Caputo type of the fractional q-derivative of order $\nu \geq 0$ is defined by

$$({}^c D_q^\nu g)(t) = (I_q^{[\nu]-\nu} D_q^{[\nu]} g)(t), \quad \nu > 0$$

where $[\nu]$ is the smallest integer greater than or equal to ν.

Definition 2.4 For any $x, y > 0$

$$B_q(x,y) = \int_0^1 t^{(x-1)}(1-qt)^{(y-1)} d_q t$$

is called q-beta function and we recall the relation

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}.$$

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Lemma 2.5 Assume $\nu, \gamma \geq 0$ and let g be a function defined on the interval [0,1]. Then

$$(i)(I_q^{\gamma} I_q^{\nu} g)(t) = (I_q^{\nu+\gamma} g)(t) \quad and$$
$$(ii)(D_q^{\nu} I_q^{\nu} g)(t) = g(t).$$

Lemma 2.6 Let $\nu > 0$. Then, the following result holds:

$$(I_q^{\nu c} D_q^{\nu} g)(t) = g(t) - \Sigma_{k=0}^{[\nu]-1} \frac{t^k}{\Gamma_q(k+1)} (D_q^k g)(0).$$

Lemma 2.7 Assume $\nu \geq 0$ and $n \in \mathbb{N}$. Then, the following equality holds:

$$\left((I_q^{\nu} D_q^n g)(t) = D_q^n I_q^{\nu} g(t) - \Sigma_{k=0}^{[\nu]-1} \frac{t^{\nu-n+k}}{\Gamma_q(\nu - n + k)} (D_q^k g)(0). \right)$$

Lemma 2.8 For $\nu \in \mathbb{R}^+$, $\rho \in (-1, \infty)$, the following is valid:

$$I_q^{\nu} \left((x-a)^{(\rho)} \right) = \frac{\Gamma_q(\rho+1)}{\Gamma_q(\nu+\rho+1)} (x-a)^{(\nu+\rho)}, \quad 0 < a < x < b.$$

For $\rho = 0$, $a = 0$, applying q- integration by parts, we get

$$(I_q^{\nu} 1)(x) = \frac{1}{\Gamma_q(\nu+1)} x^{(\nu)}.$$

3. Main Results

In this section, we establish some new oscillation criteria for (1). In the following for convenience. We denote

$$Z(t) := x(t) + p(t)x(t-\tau)$$
$$\epsilon(t) := r(\sigma(t)) \int_{t_1}^{t} \frac{{}^{c}D_q^{\alpha}(\sigma(s))}{r(\sigma(s))} d_q^{\alpha}(s)$$
$$R(t) := \int_{t_0}^{t} \frac{1}{r(s)} d_q s$$
$$\delta(t) := \int_{\rho(t)}^{\infty} \frac{1}{r(s)} d_q^{\alpha}(s)$$

Theorem 3.1 Assume that ${}^{c}D_q^{\alpha}(p(t)) \geq 0$, and there exists $\rho \in c_q^1([t_0, \infty), \mathbb{R})$.such that $\rho(t) \geq t$, ${}^{c}D_q^{\alpha}(\rho(t)) > 0$, $\sigma(t) = \rho(t) - \tau$. If for all sufficiently large t_1, and for

1*nagajothi006@gmail.com, 2ovsadha@gmail.com, 3mahahtc7409@gmail.com

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all constants $M > 0$, $L > 0$ one has,

$$\int^{\infty}\left[R^{\gamma}(\sigma(t))(1-p(\sigma(t)))^{\gamma}\phi(t)-\frac{[\gamma]M^{1-\gamma}({}^{c}D_{q}^{\alpha}(\sigma(t)))R^{\gamma-1}(\sigma(t))}{\epsilon(t)}\right]d_{q}t=\infty \tag{2}$$

and

$$\int^{\infty}\left[\phi(t)\left(\frac{1}{1+p(\rho(t))}\right)^{\gamma}\delta^{\gamma}(t)-\frac{[\gamma]({}^{c}D_{q}^{\alpha}(\rho(t)))}{L^{\gamma-1}\delta(t)r(\rho(t))}\right]d_{q}t=\infty \tag{3}$$

then (1) is oscillatory.

Proof: Suppose to the contrary that x is a nonoscillatory solution of (1).Without loss of generality we may assume that $x(t) > 0$ for all large t. The case of $x(t) < 0$ can be considered by the same method.
From (1) we can easily obtain that there exists a $t_1 \geq t_0$ such that
Case I:

$$Z(t) > 0,^{c} D_{q}^{\alpha}(Z(t)) > 0, D_{q}\left[r(t)^{c}D_{q}^{\alpha}(Z(t))\right] \leq 0 \tag{4}$$

Case II:

$$Z(t) > 0,^{c} D_{q}^{\alpha}(Z(t)) < 0, D_{q}\left[r(t)^{c}D_{q}^{\alpha}(Z(t))\right] \leq 0 \tag{5}$$

In case I holds. We have that $\sigma(t) \leq t$

$$r(t)^{c}D_{q}^{\alpha}(Z(t)) \leq r(\sigma(t))^{c}D_{q}^{\alpha}(\sigma(t)), \qquad t \geq t_1 \tag{6}$$

From the definition of Z, we have

$$\begin{aligned} z(t) &= x(t)+p(t)x(t-\tau) \\ x(t) &\geq (1-p(t))z(t) \end{aligned} \tag{7}$$

Define,

$$W(t)=R^{\gamma}(\sigma(t))\frac{r(t)^{c}D_{q}^{\alpha}(Z(t))}{(Z(\sigma(t)))^{\gamma}}, \qquad t \geq t_1. \tag{8}$$

Then $W(t) > 0$ for $t \geq t_1$. From (1),(7) and (8), we obtain

$$D_{q}W(t) \leq \frac{[\gamma]R^{\gamma-1}(\sigma(t))^{c}D_{q}^{\alpha}(\sigma(t))}{r(\sigma(t))}\frac{r((t))({}^{c}D_{q}^{\alpha}(Z((t)))}{(z\sigma(t))^{\gamma}}+R^{\gamma}(\sigma(t))$$

$$\begin{aligned}
&\left[\frac{D_q[r(t)^c D_q^\alpha z(t)](z(\sigma(t)))^\gamma - r(t)^c D_q^\alpha(Z((t))[D_q(z\sigma(t))]^\gamma}{Z(\sigma(t))^\gamma Z(\sigma(qt))^\gamma}\right] \\
&\leq \frac{[\gamma]R^{\gamma-1}(\sigma(t))^c D_q^\alpha(\sigma(t))}{r(\sigma(t))}\frac{r((t))(^c D_q^\alpha(Z((t)))}{(z\sigma(t))^\gamma} \\
&\qquad + R^\gamma(\sigma(t))\left[\frac{-\phi(t)(x(\sigma(t)))^\gamma Z(\sigma(qt))^\gamma}{Z(\sigma(t))^\gamma Z(\sigma(qt))^\gamma}\right. \\
&\qquad \left. - \frac{r(t)^c D_q^\alpha(Z((t))[\gamma]z^{\gamma-1}(\sigma(t))D_q(z(\sigma(t)))D_q(\sigma(t))}{Z(\sigma(t))^\gamma Z(\sigma(qt))^\gamma}\right]
\end{aligned}$$

$$\begin{aligned}
D_q W(t) \leq & \frac{[\gamma]R^{\gamma-1}(\sigma(t))^c D_q^\alpha(\sigma(t))}{r(\sigma(t))}\frac{r(\sigma(t))(^c D_q^\alpha(Z(\sigma(t)))}{(z\sigma(t))^\gamma} \\
& - R^\gamma(\sigma(t))\frac{\phi(t)(1-p(\sigma(t)))^\gamma Z^\gamma(\sigma(t))}{(Z(\sigma(qt)))^\gamma} \\
& - \frac{R^\gamma(\sigma(t))r(t)^c D_q^\alpha(Z(t))[\gamma]Z^{\gamma-1}(\sigma(t))D_q Z(\sigma(t))D_q(\sigma(t))}{(Z(\sigma(t)))^\gamma(Z(\sigma(qt)))^\gamma}
\end{aligned} \tag{9}$$

By (6),(9) and ${}^c D_q^\alpha(\sigma(t)) > 0$, we get

$$\begin{aligned}
D_q W(t) \leq & \frac{[\gamma]R^{\gamma-1}(\sigma(t))(^c D_q^\alpha(\sigma(t)))}{r\sigma(t)}\frac{r(\sigma(t))(^c D_q^\alpha(Z(\sigma(t)))}{(z\sigma(t))^\gamma} \\
& - R^\gamma(\sigma(t))\frac{\phi(t)(1-p(\sigma(t)))^\gamma Z^\gamma(\sigma(t))}{(Z(\sigma(qt)))^\gamma}.
\end{aligned}$$

since $x(t)$ is positive and increasing, we see that $x(t) > x(qt)$ and this implies that $-\dfrac{1}{x^2(t)} > -\dfrac{1}{x(t)(x(qt))}$.

$$\begin{aligned}
D_q W(t) \leq & \frac{[\gamma]R^{\gamma-1}(\sigma(t))(^c D_q^\alpha(\sigma(t)))}{r(\sigma(t))}\frac{r(\sigma(t))(^c D_q^\alpha(Z(\sigma(t)))}{(z\sigma(t))^\gamma} \\
& - R^\gamma(\sigma(t))\phi(t)(1-p(\sigma(t)))^\gamma
\end{aligned} \tag{10}$$

From $D_q\left[r(t)^c D_q^\alpha(Z(t))\right] \leq 0$ and ${}^c D_q^\alpha(\sigma(t)) > 0$. We see that

$$\begin{aligned}
&\int_{t_1}^{t}\left(^c D_q^\alpha(Z(\sigma(s))^c D_q^\alpha(\sigma(s))\right) d_q^\alpha s = Z(\sigma(t)) - Z(\sigma(t_1)) \\
&\quad - \sum_{k=0}^{[\alpha]-1}\frac{t^k}{\Gamma q^{k+1}}(D_q^k(Z))(\sigma(t))(0) - Z(\sigma(t_1)) + \sum_{k=0}^{[\alpha]-1}\frac{t}{\Gamma q^{k+1}}(D_q^k(Z))(\sigma(t))(0)
\end{aligned}$$

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$$Z(\sigma(t_1)) + \int_{t_1}^{t} \left({}^cD_q^\alpha(Z(\sigma(s))^cD_q^\alpha(\sigma(s))\right) d_q^\alpha s = Z(\sigma(t)) - \sum_{k=0}^{[\alpha]-1} \frac{1}{\Gamma q^{k+1}} t^K (D_q^k Z)(\sigma(t))(0) + t_1^K (D_q^K Z)(\sigma(t))(0)$$

$$\text{That is,} \quad Z(\sigma(t)) \geq \epsilon(t)^c D_q^\alpha Z(\sigma(t)). \tag{11}$$

Thus by (10)and (11), we obtain

$$D_qW(t) \leq \frac{[\gamma]R^{\gamma-1}(\sigma(t))({}^cD_q^\alpha(\sigma(t)))}{r(\sigma(t))} \frac{r(\sigma(t))({}^cD_q^\alpha(Z(\sigma(t)))}{(Z\sigma(t))(Z(\sigma(t)))^{\gamma-1}} - R^\gamma(\sigma(t))(1-p(\sigma(t)))^\gamma \phi(t).$$

$$D_qW(t) \leq \frac{[\gamma]R^{\gamma-1}(\sigma(t))({}^cD_q^\alpha(\sigma(t)))}{\epsilon(t)} M^{1-\gamma} - R^\gamma(\sigma(t))(1-p(\sigma(t)))^\gamma \phi(t). \tag{12}$$

Where $M = Z(\sigma(t))$. Integrating (12) from t_1 to t. We get $0 < W(t)$.
$W(t) \leq W(t_1) - \int_{t_1}^{t} [R^\gamma(\sigma(s))(1-P(\sigma(s)))^\gamma \phi(s)$

$$- \frac{[\gamma]M^{\gamma-1}({}^cD_q^\alpha(\sigma(s)))R^{\gamma-1}(\sigma(s))}{\epsilon(t)} \Bigg] d_q s \tag{13}$$

Letting $t \to \infty$ in (13) we get a contradiction with (2).
In case II holds. We define the function V by

$$V(t) = \frac{r(t)^cD_q^\alpha Z(t))}{Z^\gamma(\rho(t))}, \qquad t \geq t_1 \tag{14}$$

Then $V(t) < 0$ for $t \geq t_1$. Nothing $D_q\left[r(t)^cD_q^\alpha Z(t))\right] \leq 0$. $\left[r(t)^cD_q^\alpha Z(t))\right]$ is non-increasing so we have,

$$r(s)^cD_q^\alpha Z(s)) \leq r(t)^cD_q^\alpha Z(t)), \quad s \leq t \tag{15}$$

Dividing (15) by $r(s)$

$$^cD_q^\alpha Z(s) \leq \frac{r(t)^cD_q^\alpha Z(t))}{r(s)}$$

and integrating it from $\rho(t)$ to l we obtain,

$$\begin{aligned}\int_{\rho(t)}^{l}\left({}^{c}D_q^{\alpha}Z(s)\right)d_q^{\alpha}s &= Z(l)-Z(\rho(t))-\sum_{K=0}^{[\alpha]-1}\frac{l^k}{\Gamma(q^{K+1})}(D_q^kZ)(l)(0)-Z(\rho(t))\\ &\quad+\sum_{K=0}^{[\alpha]-1}\frac{(\rho t)^k}{\Gamma(q^{K+1})}(D_q^kZ)(\rho(t))(0)\\ &= r(t)^cD_q^{\alpha}Z(t)\int_{\rho(t)}^{l}\frac{1}{r(s)}d_q^{\alpha}s.\\ Z(l) &\le Z(\rho(t))+r(t)^cD_q^{\alpha}Z(t)\int_{\rho(t)}^{l}\frac{1}{r(s)}d_q^{\alpha}s.\end{aligned}$$

Letting $l\to\infty$

$$0\le Z(\rho(t))+r(t)^cD_q^{\alpha}(\delta(t)),\quad t\ge t_1$$

That is,

$$\begin{aligned}\frac{r(t)^cD_q^{\alpha}Z(t)(\delta(t))}{Z(\rho(t))}&\ge-\frac{Z(\rho(t))}{Z(\rho(t))}\\ \frac{r(t)^cD_q^{\alpha}Z(t)(\delta(t))}{Z(\rho(t))}&\ge-1.\end{aligned}$$

Thus,

$$\frac{-r(t)^cD_q^{\alpha}Z(t)\left[(-r(t)^cD_q^{\alpha}Z(t))\right]^{\gamma-1}(\delta^{\gamma}(t))}{(Z(\rho(t)))^{\gamma}}\le 1.$$

So by $(-r(t)^cD_q^{\alpha}Z(t))>0$ and (14) we have,

$$\begin{aligned}\frac{-V(t)L^{\gamma-1}\delta^{\gamma}(t)}{V(t)\delta^{\gamma}(t)}&\le\frac{1}{V(t)\delta^{\gamma}(t)}\\ L^{\gamma-1}&\le\frac{1}{V(t)\delta^{\gamma}(t)}\end{aligned}$$

$$\frac{1}{L^{\gamma-1}}\le V(t)\delta^{\gamma}(t)\le 0,\quad t\ge t_1 \tag{16}$$

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Where $L = -r(t)^c D_q^\alpha Z(t)$. Differentiating (14), we get

$$\begin{aligned} D_q V(t) &= \frac{D_q \left[r(t)^c D_q^\alpha Z(t) \right] Z^\gamma(\rho(t)) - r(t)^c D_q^\alpha Z(t) D_q(Z^\gamma(\rho(t))}{Z^\gamma(\rho(t)) Z^\gamma(\rho(qt))} \\ &= \frac{-\phi(t)(x(\sigma(t)))^\gamma Z^\gamma(\rho(t)) - r(t)^c D_q^\alpha Z(t) D_q(Z^\gamma(\rho(t)))}{Z^\gamma(\rho(t)) Z^\gamma(\rho(qt))}. \end{aligned}$$

$$D_q V(t) = -\phi(t) \frac{(x^\gamma(\sigma(t)))}{Z^\gamma(\rho(qt))} - \frac{[\gamma] r(t)^c D_q^\alpha Z(t) Z^{\gamma-1}(\rho(t)) D_q Z(\rho(t)) D_q(\rho(t))}{Z^\gamma(\rho(t)) Z^\gamma(\rho(qt))}. \quad (17)$$

Nothing that ${}^c D_q^\alpha p(t) \geq 0$.

$${}^c D_q^\alpha Z(t) =^c D_q^\alpha x(t) +^c D_q^\alpha x(t) \leq 0 \quad \text{for } t \geq t_1.$$

Hence by $(\sigma(t)) \leq (\rho(t)) - (\tau)$,we have

$$\begin{aligned} \frac{x^\gamma(\sigma(t))}{Z^\gamma(\rho(t))} &= \left(\frac{x(\sigma(t))}{x(\rho(t)) + p(\rho(t)) x(\rho(t) - \tau)} \right)^\gamma \\ &= \left(\frac{1}{\dfrac{x(\rho(t))}{x(\sigma(t))} + p(\rho(t) \dfrac{x(\rho(t)) - \tau}{x(\sigma(t))}} \right)^\gamma, \sigma(t) \leq \rho(t) - \tau. \\ \frac{x^\gamma(\sigma(t))}{Z^\gamma(\rho(t))} &\geq \left(\frac{1}{\dfrac{x(t)}{x(t)} + p(\rho(t)) \dfrac{x(\sigma(t))}{x(\sigma(t))}} \right)^\gamma \geq \left(\frac{1}{1 + p(\rho(t))} \right)^\gamma. \end{aligned}$$

Thus by (14) and (17) we get

$$D_q V(t) \leq -\phi(t) \left(\frac{1}{1 + p(\rho(t))} \right)^\gamma.$$

that is,

$$D_q V(t) + \phi(t) \left(\frac{1}{1 + p(\rho(t))} \right)^\gamma \leq 0, \quad t \geq t_1. \quad (18)$$

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and multiply (18) by $\delta^\gamma(t)$ and integrating from t_1 to t implies that

$$\delta^\gamma(t)V(t) - \delta^\gamma(t_1)V(t_1) + [\gamma]\int_{t_1}^t \frac{1}{r(\rho(s))} D_q(\rho(s))\delta^{\gamma-1}(s)V(s)d_qs$$
$$+ \int_{t_1}^t \left(\frac{1}{1+p(\rho(s))}\right)^\gamma \phi(s)\delta^\gamma(s)d_qs \leq 0. \qquad (19)$$

Therefore it follows from (16) and 19 that

$$\delta^\gamma(t)V(t) - \delta^\gamma(t_1)V(t_1) - \left[\int_{t_1}^t \phi(s)\left(\frac{1}{1+p(\rho(s))}\right)^\gamma \delta^\gamma(s) - \frac{[\gamma]D_q(\rho(s))\delta^{\gamma-1}(s)\delta^\gamma(s)V(s)}{\delta(s)r(\rho(s))}\right] d_qs.$$

$$\delta^\gamma(t)V(t) - \delta^\gamma(t_1)V(t_1) - \left[\int_{t_1}^t \phi(s)\left(\frac{1}{1+p(\rho(s))}\right)^\gamma \delta^\gamma(s) - \frac{[\gamma]^c D_q^\alpha(\rho(s))}{L^{\gamma-1}\delta(s)r(\rho(s))}\right] d_qs.$$

Letting $t \to \infty$ in the above inequality by (3). We get a contradiction with16.

Theorem 3.2 Assume that ${}^cD_q^\alpha(p(t)) \geq 0$, and there exists $\rho \in c_q^1([t_0,\infty),\mathbb{R})$,such that $\rho(t) \geq t$, ${}^cD_q^\alpha(\rho(t)) > 0$, $\sigma(t) = \rho(t) - \tau$.If for all sufficiently large t_1, and for all constants $M > 0$ such that (2) holds and

$$\int^\infty \left[\phi(t)\left(\frac{1}{1+p(\rho(t))}\right)^\gamma \delta^{\gamma+1}(t)\right] d_qt = \infty. \qquad (20)$$

then (1) is oscillatory.

Proof: Suppose to the contrary that x is a nonoscillatory solution of (1). Without loss of generality we may assume that $x(t) > 0$ for all large t. The case of $x(t) < 0$ can be considered by the same method. From (1) we can easily obtain that there exists a $t_1 \geq t_0$ such that (4) or (5) holds. If (4) holds. Proceeding as in the proof of Theorem (3.1). We obtain a contradiction with (2). If case II holds. We proceed as in the proof of Theorem (3.1) then we get (16) and (18). Multiplying (18) by $\delta^\gamma(t)$,

$$\delta^{\gamma+1}(t)D_qV(t) + \delta^{\gamma+1}(t)\phi(t)\left(\frac{1}{1+p(\rho(t))}\right)^\gamma \leq 0, \qquad t \geq t_1$$

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and integrating from t_1 to t implies that

$$\delta^{\gamma+1}(t)V(t) - \delta^{\gamma+1}(t_1)V(t_1) + [\gamma+1]\int_{t_1}^{t} \frac{1}{r(\rho(s))}{}^{c}D_q^{\alpha}(\rho(t))\delta^{\gamma}(s)V(s)d_qs$$
$$+\int_{t_1}^{t}\phi(s)\left(\frac{1}{1+p(\rho(s))}\right)^{\gamma}\delta^{\gamma+1}(s)d_qs \leq 0. \qquad (21)$$

In view of (16) we have,

$$-V(t)\delta^{\gamma+1}(t) \leq \frac{1}{L^{\gamma-1}}\delta(t) < \infty, \quad t\to\infty,$$

and consider

$$\int_{t_1}^{t} -\frac{1}{r(\rho(s))}\frac{{}^{c}D_q^{\alpha}\rho(s)\delta^{\gamma+1}(s)V(s)}{\delta(s)}d_q^{\alpha}s.$$
$$\leq \frac{1}{L^{\gamma-1}}\delta(s)\int_{t_1}^{t}\frac{{}^{c}D_q^{\alpha}\rho(s)}{r(\rho(s))\delta(s)}d_q^{\alpha}s.$$
$$\leq \frac{1}{L^{\gamma-1}}\int_{\rho(t_1)}^{\rho(t)}\frac{1}{r(u)}d_q^{\alpha}s.$$

Therefore by the above inequality letting $t\to\infty$ in (21). We obtain

$$\int^{\infty}\phi(t)\left(\frac{1}{1+p(\rho(t))}\right)^{\gamma}\delta^{\gamma+1}(t)d_qt \leq \infty,$$

which contradiction (20).

Theorem 3.3 Assume that ${}^{c}D_q^{\alpha}(p(t)) \geq 0$, and there exists $\rho \in c_q^1([t_0,\infty),\mathbb{R})$, such that $\rho(t)\geq t$, ${}^{c}D_q^{\alpha}(\rho(t)) > 0$, $\sigma(t) = \rho(t)-\tau$. If for all sufficiently large t_1, and for all constants $M>0$ such that (2) holds and

$$\int_{t_1}^{\infty}\frac{1}{r(v)}\int_{t_1}^{v}\phi(u)\left(\frac{1}{1+p(\rho(u))}\right)^{\gamma}\delta^{\gamma}(u)d_qu \;\; d_qv = \infty, \qquad (22)$$

then (1) is oscillatory.

Proof: Suppose to the contrary that u is a nonoscillatory solution of (1).Without loss of generality we may assume that $x(t)>0$ for all large t. The case of $x(t)<0$ can be considered by the same method.
From (1) we can easily obtain that there exists a $t_1 \geq t_0$ such that (4) or (5) holds.

If (4) holds. Procedding as in the proof of Theorem (3.1). We obtain a contradiction with (2).
If case II holds. We proceed as in the proof of Theorem (3.1) then we get (15). Dividing (15) by $r(s)$

$$^{c}D_q^{\alpha}Z(s) \leq \frac{r(t)cD_q^{\alpha}Z(t)}{r(s)}.$$

and integrating it from $\rho(t)$ to l, letting $l \to \infty$ yields.

$$\begin{aligned}
Z(\rho(t)) - Z(l) &\geq -r(t)^{c}D_q^{\alpha}Z(t)\int_{\rho(t)}^{\infty}\frac{1}{r(s)}d_qs \\
&\geq -r(t)^{c}D_q^{\alpha}Z(t)\delta(t) \\
Z(\rho(t)) - Z(l) &\geq: a\delta(t).
\end{aligned}$$

by (1) We have

$$D_q\left[-r(t)^{c}D_q^{\alpha}Z(t)\right] = \phi(t)x^{\gamma}(\sigma(t)).$$

Noticing that $^{c}D_q^{\alpha}P(t) \geq o$, We see that $^{c}D_q^{\alpha}x(t) \leq 0$ for $t \geq t_1$, so by $\sigma(t) \leq \rho(t) - \tau$, We get

$$\begin{aligned}
\frac{x(\sigma(t))}{Z(\rho(t))} &= \frac{x(\sigma(t))}{x(\rho(t)) + p(\rho(t))x(\rho(t)-\tau} \\
&\geq \frac{1}{1+p(\rho(t))}.
\end{aligned}$$

Hence we obtain

$$D_q\left[-r(t)^{c}D_q^{\alpha}Z(t)\right] \geq a^{\gamma}\phi(t)\left(\frac{1}{1+p(\rho(t))}\right)^{\gamma}\delta^{\gamma}(t).$$

Integrating the above inequality from t_1 to t, we have

$$\begin{aligned}
-r(t)^{c}D_q^{\alpha}Z(t) &\geq -r(t_1)^{c}D_q^{\alpha}Z(t_1) + a^{\gamma}\int_{t_1}^{t}\phi(u)\left(\frac{1}{1+p(\rho(t))}\right)^{\gamma}\delta^{\gamma}(u)d_qu. \\
&\geq a^{\gamma}\int_{t_1}^{t}\phi(u)\left(1+P(\rho(t))\right)^{\gamma}\delta^{\gamma}(u)d_qu.
\end{aligned}$$

Integrating the above inequality from t_1 to t, we obtain

$$z(t_1) - Z(t) \geq a^\gamma \int_{t_1}^{t} \frac{1}{r(v)} \int_{t_1}^{v} \phi(u) \left(\frac{1}{1 + p(\rho(u))} \right)^\gamma \delta^\gamma(u) d_q u \ \ d_q v.$$

Which contradicts (22).

4. Conclusion

In this paper, we have obtained some oscillation results for the fractional order Emden-Fowler quantum difference equation using generalized Riccati technique. In this paper is q-analog of [6]. Our results are new.

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1*nagajothi006@gmail.com, 2ovsadha@gmail.com, 3mahahtc7409@gmail.com

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[1*]nagajothi006@gmail.com, [2]ovsadha@gmail.com, [3]mahahtc7409@gmail.com