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Oscillation and Nonoscillation of Solutions of Generalized Nonlinear Difference Equation of Second Order

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Abstract

In this paper, the authors discuss oscillation and nonoscillation of solutions of the generalized nonlinear difference equation

$$\Delta_\ell(p(k)\Delta u(k)) + f(k)F(k, u(k), \Delta_\ell u(k)) = g(k, u(k), \Delta_\ell u(k)), \tag{1}$$

$k \in [a, \infty)$, where the functions p, f, F and g are defined in their domain of definition and ℓ is a positive real. Further, $p(k) > 0$ for all $k \in [a, \infty)$ for some $a \in [0, \infty)$ and for all $j = k - a - \left[\frac{k-a}{\ell}\right]\ell$, $R_{a+j,k} \to \infty$, where $R_{t+j,k} = \sum\limits_{n=n_0}^{k-1} \frac{1}{p(n)}$, $t \in [a, \infty)$ and $k \in \mathbb{N}_\ell(t + j + \ell)$.

Key words: Generalized difference equation, generalized difference operator, oscillation and nonoscillation.

AMS Subject classification: 39A12.

1. Introduction

The basic theory of difference equations is based on the operator Δ defined as $\Delta u(k) = u(k+1) - u(k),\ k \in \mathbb{N} = \{0, 1, 2, 3, \cdots\}$. Eventhough many authors ([1], [14]-[16]) have suggested the definition of Δ as

$$\Delta u(k) = u(k + \ell) - u(k),\ k \in \mathbb{R},\ \ell \in \mathbb{N}(1), \tag{2}$$

no significant progress has taken on this line. But recently, when we took up the definition of Δ as given in (2) we developed the theory of difference equations in a different direction (see [7]-[8]). For convenience, we labelled the operator Δ defined by (2) as Δ_ℓ and by defining its inverse Δ_ℓ^{-1}, many interesting results and applications in number theory (See [7],[10]-[13]) were obtained. By extending the study related

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to sequences of complex numbers and ℓ to be real, some new qualitative properties like rotatory, expanding, shrinking, spiral and weblike were analysed for the solutions of difference equations involving Δ_ℓ. The results obtained using Δ_ℓ can be found in ([7]-[13]).

In [6], John R. Graef worked on Oscillation, nonoscillation, and growth of solutions of nonlinear functional differential equations of arbitrary order and Blazej Szmanda [3] obtained the discrete analogous of [6]. In [3] the author considered $\ell = 1$ and $k \in \mathbb{N}(a)$ for an integer a but, in this paper the theory is extended for all real $k \in [a, \infty)$ and for any real ℓ and oscillation and nonoscillation of solutions of the generalized nonlinear difference equation (1) is discussed. The results of this paper generalize those of ([4, 5, 11]).

Throughout this paper, we use the following notations.
(i) $\mathbb{N} = \{0, 1, 2, 3, \dots\},\ \mathbb{N}(a) = \{a, a+1, a+2, \dots\}$,
(ii) $\mathbb{N}_\ell(j) = \{j, j+\ell, j+2\ell, \dots\}$.
(iii) $\lceil x \rceil$ denotes upper integer part of x and $[X]$ denotes the integer part of X.

2. Preliminaries

In this section, we present some basics already existing in the literature which is useful for further discussion.

Definition 2.1 [7] Let $u(k),\ k \in [0, \infty)$ be a real or complex valued function and $\ell \in (0, \infty)$. Then, the generalized difference operator Δ_ℓ is defined as

$$\Delta_\ell u(k) \ = \ u(k+\ell) - u(k). \tag{3}$$

Similarly, the generalized difference operator of the r^{th} order is defined as

$$\Delta_\ell^r u(k) = \underbrace{\Delta_\ell(\Delta_\ell(\dots(\Delta_\ell u(k))\dots))}_{r\ times}. \tag{4}$$

Definition 2.2 [7] Let $u(k),\ k \in [0, \infty)$ be a real or complex valued function and $\ell \in (0, \infty)$. Then, the inverse of Δ_ℓ denoted by Δ_ℓ^{-1} is defined as follows.

$$\text{If } \Delta_\ell v(k) \ = \ u(k), \ \ \text{then } v(k) \ = \ \Delta_\ell^{-1} u(k) + c_j, \tag{5}$$

where c_j is a constant for all $k \in \mathbb{N}_\ell(j),\ j = k - \left[\frac{k}{\ell}\right] \ell$.
In general, $\Delta_\ell^{-n} u(k) = \Delta_\ell^{-1}(\Delta_\ell^{-(n-1)} u(k))$ for $n \in \mathbb{N}(2)$.

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Lemma 2.3 [7] If the real valued function $u(k)$ is defined for all $k \in [a, \infty)$, then,

$$\Delta_\ell^{-1} u(k) = \sum_{r=1}^{\left[\frac{k-a}{\ell}\right]} u(k - r\ell) + c_j, \tag{6}$$

where c_j is a constant for all $k \in \mathbb{N}_\ell(j)$, $j = k - a - \left[\frac{k-a}{\ell}\right]\ell$.

Corollary 2.4 If $\Delta_\ell v(k) = u(k)$ for $k \in [k_2, \infty)$ and $j = k - k_2 - \left[\frac{k-k_2}{\ell}\right]\ell$, then

$$v(k) - v(k_2 + j) = \sum_{r=0}^{\frac{k-k_2-j-\ell}{\ell}} u(k_2 + j + r\ell).$$

proof The proof follows by Definition 2.2, Lemma 2.4 and $c_j = v(k_2 + j)$.

Definition 2.5 [1] The solution $u(k)$ of (1) is called oscillatory if for any $k_1 \in [a, \infty)$ there exists a $k_2 \in \mathbb{N}_\ell(k_1)$ such that $u(k_2)u(k_2 + \ell) \leq 0$. The difference equation itself is called oscillatory if all its solutions are oscillatory. If the solution $u(k)$ is not oscillatory then it is said to be nonoscillatory (i.e. $u(k)u(k + \ell) > 0$ for all $k \in [k_1, \infty)$).

In order to prove the main results one or more of the following conditions have been used.
(c_1) $f(k) \geq 0$ for all $k \in [a, \infty)$,
(c_2) there exists a constant M_1 such that $F(k, u, v) \geq M_1$,
(c_3) there exists a constant M_2 such that $F(k, u, v) \leq M_2$
(c_4) there exists a constant $M > 0$ such that $|F(k, u, v)| \leq M$,
(c_5) there exists a function $\phi(k)$ such that $g(k, u, v) \geq \phi(k)$,
(c_6) there exists a function $\psi(k)$ such that $g(k, u, v) \leq \psi(k)$,
(c_7) $F(k, u, v)$ is bounded from above if u is bounded,
(c_8) $F(k, u, v)$ is bounded from below if u is bounded,
(c_9) $uF(k, u, v) \geq 0$,
(c_{10}) $uF(k, u, v) \leq 0$,
(c_{11}) there exist functions $m(k)$ and $n(k)$ such that $m(k) \leq F(k, u, v) \leq n(k)$,
(c_{12}) there exists a nonnegative real function $B(k)$ such that $|g(k, u, v)| \leq B(k)$,
(c_{13}) there exists a nonnegative real function $m(k)$ such that $|F(k, u, v)| \leq m(k)|u|$.

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3. Main Results

Nonoscillation Results

In this section, we present conditions for the nonoscillation of equation (1).

Theorem 3.1 Suppose that conditions (c_1),(c_3) and (c_5) hold and for every constant $C > 0$,

$$\liminf_{k\to\infty} \Big(\sum_{n=n_0}^{k-1} \frac{1}{p(n)} \sum_{r=0}^{\frac{k-\ell-a-j}{\ell}} (\phi(a+j+r\ell) - M_2 f(a+j+r\ell)) - CR_{a+j,k} \Big) > 0, \quad (7)$$

for all $0 \le j < \ell$ and $k \in \mathbb{N}_\ell(a+j+\ell)$. Then, all solutions of (1) are eventually positive.
proof Let $u(k)$ be a solution of (1). Applying conditions (c_1), (c_3) and (c_5), we obtain

$$\Delta_\ell(p(k)\Delta u(k)) \ge \phi(k) - M_2 f(k), \ k \in [a, \infty).$$

Therefore, it follows from Corollary 2.4 that, for all $0 \le j < \ell$,

$$\Delta u(k) \ge \frac{p(a+j)}{p(k)} \Delta u(a+j) + \frac{1}{p(k)} \sum_{r=0}^{\frac{k-\ell-a-j}{\ell}} (\phi(a+j+r\ell) - M_2 f(a+j+r\ell)),$$

where $k \in \mathbb{N}_\ell(a+j+\ell)$ and $j = k - a - \left[\frac{k-a}{\ell}\right]\ell$. Again applying Corollary 2.4, we get

$$u(k) \ge u(a) + p(a+j)\Delta u(a+j) \sum_{n=n_0}^{k-1} \frac{1}{p(n)}$$
$$+ \sum_{n=n_0}^{k-1} \frac{1}{p(n)} \sum_{r=0}^{\frac{k-\ell-a-j}{\ell}} (\phi(a+j+r\ell) - M_2 f(a+j+r\ell)).$$

Now, in view of the conditions on $p(k)$, there exist constants $C > 0$ and $k_1 \in [a+j+2\ell, \infty)$ such that

$$-CR_{a+j,k} \le u(a) + p(a+j)\Delta u(a+j) \sum_{n=n_0}^{k-1} \frac{1}{p(n)}$$
$$\le CR_{a+j,k}, \ k \in [k_1, \infty). \quad (8)$$

Hence, from (7) we have

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$$\liminf_{k\to\infty} u(k) \geq \liminf_{k\to\infty} \Big(\sum_{n=n_0}^{k-1} \frac{1}{p(n)} \sum_{r=0}^{\frac{k-\ell-a-j}{\ell}} (\phi(a+j+r\ell) - M_2 f(a+j+r\ell)) - CR_{a+j,k} \Big) > 0.$$

Thus, $u(k)$ is eventually positive.

The proofs of the following results are similar to that of Theorem 3.1 and therefore are omitted.

Theorem 3.2 Suppose that conditions (c_1), (c_2) and (c_6) hold and for every constant $C > 0$,

$$\limsup_{k\to\infty} \Big(\sum_{n=n_0}^{k-1} \frac{1}{p(n)} \sum_{r=0}^{\frac{k-\ell-a-j}{\ell}} (\psi(a+j+r\ell) - M_1 f(a+j+r\ell)) + CR_{a+j,k} \Big) < 0. \quad (9)$$

Then, all solutions of (1) are eventually negative.

Theorem 3.3 Suppose that conditions (c_1),(c_5) and (c_7) hold and for every constants $C_1, C_2 > 0$,

$$\liminf_{k\to\infty} \Big(\sum_{n=n_0}^{k-1} \frac{1}{p(n)} \sum_{r=0}^{\frac{k-\ell-a-j}{\ell}} (\phi(a+j+r\ell) - C_1 f(a+j+r\ell)) - C_2 R_{a+j,k} \Big) > 0. \quad (10)$$

Then, all bounded solutions of (1) are eventually positive.

Theorem 3.4 Suppose that conditions (c_1), (c_6) and (c_8) hold and for every constants $C_1, C_2 > 0$,

$$\limsup_{k\to\infty} \Big(\sum_{n=n_0}^{k-1} \frac{1}{p(n)} \sum_{r=0}^{\frac{k-\ell-a-j}{\ell}} (\psi(a+j+r\ell) + C_1 f(a+j+r\ell)) + C_2 R_{a+j,k} \Big) < 0. \quad (11)$$

Then, all bounded solutions of (1) are eventually negative.

Theorem 3.5 Suppose that conditions (c_4) and (c_5) hold and for every constant $C > 0$,

$$\liminf_{k\to\infty} \Big(\sum_{n=n_0}^{k-1} \frac{1}{p(n)} \sum_{r=0}^{\frac{k-\ell-a-j}{\ell}} (\phi(a+j+r\ell) - M f(a+j+r\ell)) - CR_{a+j,k} \Big) > 0. \quad (12)$$

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Then, all solutions of (1) are eventually positive.

Theorem 3.6 Suppose that conditions (c_4) and (c_6) hold and for every constant $C > 0$,

$$\lim_{k\to\infty} \sup \Big(\sum_{n=n_0}^{k-1} \frac{1}{p(n)} \sum_{r=0}^{\frac{k-\ell-a-j}{\ell}} (\psi(a+j+r\ell) + M|f(a+j+r\ell)|) + CR_{a+j,k} \Big) < 0. \quad (13)$$

Then, all solutions of (1) are eventually negative.

Theorem 3.7 Suppose that conditions (c_5),(c_7) and (c_8) hold and for every constants $C_1, C_2 > 0$,

$$\lim_{k\to\infty} \inf \Big(\sum_{n=n_0}^{k-1} \frac{1}{p(n)} \sum_{r=0}^{\frac{k-\ell-a-j}{\ell}} (\phi(a+j+r\ell) - C_1|f(a+j+r\ell)|) - C_2R_{a+j,k} \Big) > 0. \quad (14)$$

Then, all bounded solutions of (1) are eventually positive.

Theorem 3.8 Suppose that conditions (c_6) - (c_8) hold and for every constants $C_1, C_2 > 0$,

$$\lim_{k\to\infty} \sup \Big(\sum_{n=n_0}^{k-1} \frac{1}{p(n)} \sum_{r=0}^{\frac{k-\ell-a-j}{\ell}} (\psi(a+j+r\ell) + C_1|f(a+j+r\ell)|) + C_2R_{a+j,k} \Big) < 0. \quad (15)$$

Then, all bounded solutions of (1) are eventually negative.

Oscillation Results

In this section we present conditions for oscillation of (1).

Theorem 3.9 Suppose that $f(k) \equiv 1$ and conditions (c_5), (c_6) and (c_{11}) hold. Further, let for every constant $C > 0$ and all large $s \in [a, \infty)$,

$$\lim_{k\to\infty} \inf \left(\sum_{n=n_0}^{k-1} \frac{1}{p(n)} \sum_{r=0}^{\frac{k-\ell-s-j}{\ell}} (\psi(s+j+r\ell) - m(s+j+r\ell)) + CR_{s+j,k} \right) < 0. \quad (16)$$

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and

$$\lim_{k\to\infty}\sup\left(\sum_{n=n_0}^{k-1}\frac{1}{p(n)}\sum_{r=0}^{\frac{k-\ell-s-j}{\ell}}(\phi(s+j+r\ell)-n(s+j+r\ell))-CR_{s+j,k}\right)>0. \quad (17)$$

Then, the generalized difference equation (1) is oscillatory.

proof Let $u(k)$ be a nonoscillatory solution of (1), say $u(k)>0$ for all $s\le k\in[a,\infty)$. Then, from (1) we have

$$\phi(k)-n(k)\le\Delta_\ell(p(k)\Delta u(k))\le\psi(k)-m(k),\ k\in[s,\infty).$$

Now, following as in Theorem 3.1 we obtain

$$\sum_{n=n_0}^{k-1}\frac{1}{p(n)}\sum_{r=0}^{\frac{k-\ell-s-j}{\ell}}(\phi(s+j+r\ell)-n(s+j+r\ell))-CR_{s+j,k}\le u(k)$$
$$\le\sum_{n=n_0}^{k-1}\frac{1}{p(n)}\sum_{r=0}^{\frac{k-\ell-s-j}{\ell}}(\psi(s+j+r\ell)-m(s+j+r\ell))+CR_{s+j,k}.$$

Condition (16) then yields a contradiction to the assumption that $u(k)>0$ for all $k\in[s,\infty)$. A similar proof holds if $u(k)<0$ for all $k\in[s,\infty)$. This completes the proof.

Theorem 3.10 Suppose that $f(k)\equiv 1$ and conditions (c_5), (c_6) and (c_9) hold. Further, let for every constant $C>0$ and all large $s\in[a,\infty)$

$$\lim_{k\to\infty}\inf\left(\sum_{n=n_0}^{k-1}\frac{1}{p(n)}\sum_{r=0}^{\frac{k-\ell-s-j}{\ell}}\psi(s+j+r\ell)+CR_{s+j,k}\right)<0. \quad (18)$$

and

$$\lim_{k\to\infty}\sup\left(\sum_{n=n_0}^{k-1}\frac{1}{p(n)}\sum_{r=0}^{\frac{k-\ell-s-j}{\ell}}\phi(s+j+r\ell)-CR_{s+j,k}\right)>0. \quad (19)$$

Then, the difference equation (1) is oscillatory.

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proof Let $u(k)$ be a nonoscillatory solution of (1), say $u(k) \geq 0$ for $k \in [s, \infty)(s \geq 0)$. From (1) we have

$$\Delta_\ell(p(k)\Delta u(k)) = -F(k, u(k), \Delta_\ell u(k)) + g(k, u(k), \Delta_\ell u(k)) \leq p(k) \text{ for } k \in [s, \infty)$$

Proceeding as before we observe that there exists $C > 0$ and $k_1 \in [s, \infty)$ such that an estimate of (8) holds for $k \in [k_1, \infty)$ and

$$u(k) \leq CR_{s,k} + \sum_{n=n_0}^{k-1} \frac{1}{p(n)} \sum_{r=0}^{\frac{k-\ell-s-j}{\ell}} \psi(s + j + r\ell) \text{ for } k \in [k_1, \infty).$$

Then condition (18) gives a contradiction to the assumption that $u(k) \geq 0$ for $k \in [s, \infty)$. The proof in case $u(k) \leq 0$ for $k \in [s, \infty)$ is similar.

Theorem 3.11 Suppose that $f(k) \equiv 1$ and conditions (c_5), (c_6) and (c_{10}) hold. Further, let for every constant $C > 0$ and all large $s \in [a, \infty)$

$$\liminf_{k\to\infty} \left(\sum_{n=n_0}^{k-1} \frac{1}{p(n)} \sum_{r=0}^{\frac{k-\ell-s-j}{\ell}} \psi(s + j + r\ell) + CR_{s+j,k} \right) = -\infty. \tag{20}$$

and

$$\limsup_{k\to\infty} \left(\sum_{n=n_0}^{k-1} \frac{1}{p(n)} \sum_{r=0}^{\frac{k-\ell-s-j}{\ell}} \phi(s + j + r\ell) - CR_{s+j,k} \right) = \infty. \tag{21}$$

Then, all the bounded solutions of the difference equation (1) are oscillatory.

proof Let $u(k)$ be a nonoscillatory solution of (1), say $u(k) \geq 0$ for $k \in [s, \infty)(s \geq 0)$ and assume that $u(k)$ is bounded. Arguing as in the proof of the previous theorems, we obtain the inequality

$$u(k) \geq \sum_{n=n_0}^{k-1} \frac{1}{p(n)} \sum_{r=0}^{\frac{k-\ell-s-j}{\ell}} \phi(s + j + r\ell) - CR_{s+j,k} \text{ for } k \in [k_1, \infty)(k_1 \geq s).$$

Then (21) gives a contradiction to the boundedness of $u(k)$. A similar argument treats the case of an eventually nonpositive solution.

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