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# Oscillation Criteria for Generalized First-Order Systems of Linear Difference Equations

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**Abstract**

In this article, we obtain conditions for the oscillation of vector solutions to the generalized first-order systems of linear difference equations $u(k+\ell) = a(k)u + b(k)v$, $v(k+\ell) = c(k)u + d(k)v$ and $u(k+\ell) = a(k)u + b(k)v + f_1(k)$ $v(k+\ell) = c(k)u + d(k)v + f_2(k)$ where $a(k), b(k), c(k), d(k)$ and $f_i(k), i = 1, 2$ are real valued functions defined for $k \geq 0$.

**Key words:** Generalized, Linear, Vector Analysis.

**AMS classification:** 39A10, 39A11, 39A12

## 1. Introduction

Difference equations represent a fascinating mathematical area on its own as well as a rich field of the applications in such diverse disciplines as population dynamics, operations research, ecology, biology etc. For general background as difference equations with many examples from diverse fields, one can refer to [7].

The theory of difference equations is based on the operator $\Delta$ defined as

$$\Delta u(k) = u(k + 1) - u(k), k \in \mathbb{N} = \{0, 1, 2, \ldots\}.$$

Even though many authors [7],[15] have suggested the definition of $\Delta$ as

$$\Delta u(k) = u(k + \ell) - u(k), \ k \in [0, \infty), \ \ell \in (0, \infty), \tag{1}$$

no significant progress took place on this line. In [2], they took up the definition of $\Delta$ as given in (1), and developed the theory of difference equations in a different direction. For convenience, we labeled the operator $\Delta$ defined by (1) as $\Delta_\ell$ and by defining its inverse $\Delta_\ell^{-1}$, many interesting results in numerical methods were obtained.

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During the last several years many research papers on the oscillatory behavior of solutions of difference equations have appeared in the literature, as these equations occur as mathematical models of some real world problems ([9, 10, 15]).

Consider the system of $k-$equations of the form

$$U(k + \ell) = AU(k), \tag{2}$$

where $A = (a_{ij})_{n \times n}$ is a constant matrix. The characteristic equation of (2) is given by $det(\lambda I - A) = 0$;that is,

$$\lambda^n + (-1)^n b_1 \lambda^{n-1} + \cdots + (-1)^n b_n = 0, \tag{3}$$

where $b_n = det A$. If $n$ is odd, then from the theory of algebraic equations (see e.g. [8]), it follows that (3) admits at least one real root $\lambda_1$ such that the sign of $\lambda_1$ is opposite to that of the last term, namely $(1)^n b_n$. Hence we have the following result.

## 2. Main Results

**Theorem 2.1** Let $n$ be odd. If $det\ A < 0$, then (2) admits at least one oscillatory solution; if $det\ A > 0$, then (2) admits at least one nonoscillatory solution.

Proof: When $det\ A < 0$, we find a real root $\lambda_1$ of (3) such that $\lambda_1 < 0$ and $U(k) = (\lambda_1)^k C$, where $C = (C_1 C_2 \cdots C_k)^T$ is a column vector of constants. Thus $U(k)$ is oscillatory. Similarly for $det\ A > 0$.

**Remark 2.2** If $det\ A = 0$, then (2) admits a nonoscillatory solution. Indeed, $det\ A = 0$, implies that $\lambda = 0$ is a solution of (3) and hence $U(k) = C$ is a solution of (2), where $C$ is a non-zero constant vector. We note that $AC = 0$ always admits a nontrivial solution.

**Theorem 2.3** Let $n$ be even. If $det\ A < 0$, then (2) admits an oscillatory solution and a nonoscillatory solution.

Proof: The proof is simple and can be obtained from the following Theorem in [8].

**Theorem 2.4** (I) Every equation of an even degree, whose constant term is negative has at least two real roots one positive and the other negative.
(II) If the equation contains only even powers of u and the coefficients are all of the same sign, then the equation has no real root; that is, all roots are complex.

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**Remark 2.5** If the last term of an even degree equation is positive, no definite conclusion can be drawn regarding the roots of the equation. If $det\ A > 0$, then no definite conclusion can be drawn regarding the oscillation of solutions of (2) when $n$ is even.

**Theorem 2.6** Let $n$ be even and $A$ be such that
$b_1 = b_3 = \cdots = b_{n-1} = 0,\ b_2 > 0, b_4 > 0 \cdots b_n > 0$. Then every component of the vector solution of )(2) is oscillatory.

Proof: The proof of the above theorem follows from the above 2.4(II). The literature on study of system of difference equations does not consider the case when $n$ is even. Therefore the present work is devoted to study the system of equations

$$u(k + \ell) = a(k)u + b(k)v$$
$$v(k + \ell) = c(k)u + d(k)v \qquad (4)$$

and the corresponding nonhomogeneous system of equations

$$u(k + \ell) = a(k)u + b(k)v + f_1(k)$$
$$v(k + \ell) = c(k)u + d(k)v + f_2(k), \qquad (5)$$

where $a(k), b(k), c(k), d(k), f_1(k), f_2(k)$ are real-valued functions defined for $k \geq k_0 \geq 0$. One may think of systems (4) and (5) as being a discrete analog of the differential systems

$$u'(t) = a(t)u + b(t)v$$
$$v'(t) = c(t)u + d(t)v \qquad (6)$$

and

$$u'(t) = a(t)u + b(t)v + f_1(t)$$
$$v'(t) = c(t)u + d(t)v + f_2(t) \qquad (7)$$

respectively, where $a, b, c, d, f_1, f_2$ are in $C(R, R)$. If $a(k) \equiv a$, $b(k) \equiv b$, $c(k) \equiv c$ and $d(k) \equiv d$, then the characteristic equation of (4) is

$$\lambda^2 - (a + c)\lambda + (ad - bc) = 0. \qquad (8)$$

We note that this equation is the same for both the systems (4) and (6). Hence the oscillatory behavior of solutions of these systems are comparable. Clearly, the

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components of the vector solution of (6) are oscillatory only if (8) has complex roots. Otherwise, it is nonoscillatory. On the other hand, the behavior of the components of the vector solution of (4) is given below.

**Proposition 2.7** Let $\lambda_1$ and $\lambda_2$ be the roots of (8). If any one of the following three conditions

1. $(a-d)^2 + 4bc < 0$,
2. $(a-d)^2 + 4bc > 0$ but $(a+d) \pm [(a-d)^2 + 4bc]^{\frac{1}{2}} < 0$,
3. $(a-d)^2 + 4bc = 0$ and $(a+d) < 0$

hold, then every component of the vector solution of (4) is oscillatory. Otherwise, there exists a nonoscillatory solution to (4).

Proof: The proof is simple and hence it is omitted.
The object of this work is to establish the sufficient conditions for the oscillation of all solutions of the systems (4) and (5). Proposition 2.7 which demonstrate the difference in the behaviour of the solutions of the systems (4)-(5) and (6)-(7) motivate us to study further for the oscillatory behaviour of solutions of (4)-(5). Furthermore, an attempt is made here to apply some of the results of [6] for the oscillatory behaviour of solutions of the systems (4) and (5).

A close observation reveals that, all most all works in difference equations / system of equations are the discrete analogue of the differential equations / system of equations see for e.g. [1, 12, 13] and the references cited therein. Agarwal and Grace [1] have discussed the oscillatory behavior of solutions of the system of equations of the form $(1)^{m+1}\Delta_\ell^m v((i)k) + \sum_{j=1}^K q(i,j)v(j)(k - \tau(j,j)) = 0, m \geq 1, \ i = 1,2,\cdots,K$ which is the discrete analogue of the functional differential equations

$$\frac{d^m}{dt^m}v(i(t)) + \sum_{j=1}^K q(i,j)v(j)(t\tau(j,j)) = 0, \ m \geq 1, \ i = 1,2,\cdots,K,$$

where $q(i,j)$ and $\tau(j,j)$ are real numbers and $\tau(j,j) > 0$. It seems that the results in [1] are the discrete analog results of the continuous case. We note that, in this work an investigation is made to study the system of equations (4)/(5) without following any results of the continuous case.

By a solution of (4)/(5) we mean a real valued vector function $U(k)$ for $k \in (0, \infty)$ which satisfies (4)/(5). We say that the solution $U(k) = [u(k), v(k)]^T$ oscillates component wise or simply oscillates if each component oscillates. Otherwise, the

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solution $U(k)$ is called non-oscillatory. Therefore a solution of (4)/(5) is non-oscillatory if it has a component which is eventually positive or eventually negative.

We need the following two results from [6] for our use in the sequel.

**Theorem 2.8** If $a(k) > 0, b(k) > 0$ and $a(k) \leq \frac{b(k+\ell)}{a(k+\ell)} + \frac{b(k)}{a(k)\ell}$ for large $k$, then $v(k+2) - a(k)v(k+\ell) + b(k)v(k)$ is oscillatory.

**Theorem 2.9** Let $0 \leq a(k) \leq 1$ and $c(k) \geq 0$. Let $f(k) = g(k+2) - g(k+\ell)$, where for each $k \geq 1$, there exists $m > k$ such that $g(k)g(m) < 0$. If
$$\sum_{k=1}^{\infty}[(1 - a(k))g^+(k+\ell) + C(k)g^+(k)] = \infty,$$
$$\sum_{k=1}^{\infty}[(1 - a(k))g^-(k+\ell) + C(k)g^-(k)] = \infty$$
then all solutions of $v(k+2) - a(k)v(k+\ell) + c(k)v(k) = f(k)$ oscillate, where $g^+(k) = \max\{g(k), 0\}$ and $g^-(k) = \max\{-g(k), 0\}$.

## 3. Oscillation for System (4)

Consider the system of equations (4) as $U(k + \ell) = A(k)U$, where $U(k) = [u(k), v(k)]^T$ and

$$A = \begin{bmatrix} a(k) & b(k) \\ c(k) & d(k) \end{bmatrix}.$$

We assume that $a(k), b(k), c(k), d(k)$ are real valued functions defined for $k \geq k_0 > 0$. Let $b(k) \neq 0$ for all $k \geq k_0$. Then it follows from (4) that

$$v(k) = \frac{u(k+\ell)}{b(k)} - \frac{a(k)}{b(k)}u(k);$$

that is, $v(k + \ell) = \frac{u(k+l)}{b(k+\ell)} - \frac{a(k+\ell)}{b(k+\ell)}u(k+\ell);$

Hence, $c(k)x(k) + d(k)v(k) = \frac{u(n+2\ell)}{b(k+\ell)}\frac{a(k+\ell)}{b(k+\ell)}u(k+\ell);$

that is,

$$u(k + 2\ell)P_1(k)u(k + \ell) + Q_1(k)u(k) = 0, \tag{9}$$

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where we define

$$P_1(k) = a(k + \ell) + \frac{d(k)b(k + \ell)}{b(k)},$$

$$Q_1(k) = \frac{b(k + \ell)}{b(k)}[a(k)d(k) - b(k)c(k)]$$

for all $k \geq k_0$. Similarly, if $c(k) \neq 0$ for all $k \geq k_0$, then

$$v(k + 2\ell) - P_2(k)v(k + \ell) + Q_2(k)v(k) = 0, \qquad (10)$$

where we define $P_2(k) = d(k + \ell) + \frac{a(k)d(k)}{c(k)}$, $Q_2(k) = \frac{c(k+\ell)}{c(k)}[a(k)d(k) - b(k)c(k)]$

**Theorem 3.1** Let $P_i(k) > 0, Q_i(k) > 0, i = 1, 2$ be such that

$$P_i(k) \leq \frac{Q_i(k + \ell)}{P_i(k + \ell)} + \frac{Q_i(k)}{P_i(k\ell)} \qquad (11)$$

for all large $k$, then every solution $U(k)$ of (4) oscillates.

Proof: Suppose, on the contrary, that $U(k)$ is a nonoscillatory solution of (4). Then there exists $k_0 > 0$ such that at least one component of $U(k)$ is nonoscil- latory for $k \geq k_0$. Let $u(k)$ be the nonoscillatory component of $U(k)$ such that $u(k)$ is eventually positive for $k \geq k_0$. Then applying Proposition 2.7, we have a contradiction to (11). Similarly, one can proceed for $v(k)$, if we assume that $v(k)$ is a nonoscillatory component of $U(k)$ for $k \geq k_0$. Hence the proof is complete.

**Remark 3.2** If (11) holds for either $i = 1$ or $i = 2$, then there could be a possibility for the existence of nonoscillatory solution. However, we are not sure about the fact. We note that (9) and (10) are non self-adjoint difference equations. Hence the above possibility may be true.

**Remark 3.3** If $P_i(k) = p_i$ and $Q_i(k) = q_i$, $i = 1, 2$ then (11) becomes $p_2 \leq 2q_i$, $i = 1, 2$. Hence the inequalities $p_2 \leq 2q_1$ and $p_2 \leq 2q_2$ reduce to $(a + d)^2 \leq 2(ad - bc)$. Thus we have the following corollary.

**Corollary 3.4** If $A(k) \equiv A$ and $(trA)2 \leq 2detA$, then (4) is oscillatory.

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**Example 3.5** Consider

$$\begin{bmatrix} u(k+\ell) \\ v(k+\ell) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} u(k) \\ v(k) \end{bmatrix} \tag{12}$$

Indeed, $tr A = 2$ and $\det A = 3$. $\lambda_1 = 1 + i\sqrt{2}$ and $\lambda_2 = 1 - i\sqrt{2}$ are two characteristic roots of the coefficient matrix $A$. Clearly,

$$\begin{aligned} u(k) &= \lambda_1^k \begin{bmatrix} 1 \\ i\sqrt{2} \end{bmatrix} \\ &= (1 + i\sqrt{2})^k \begin{bmatrix} 1 \\ i\sqrt{2} \end{bmatrix} \\ &= 3^{k/2}(\cos k\theta + i\sin k\theta) \begin{bmatrix} 1 \\ i\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 3^{k/2}(\cos k\theta + i\sin k\theta) \\ -3^{k/2}(\cos k\theta + i\sin k\theta) \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} v(k) &= \lambda_2^k \begin{bmatrix} 1 \\ -i\sqrt{2} \end{bmatrix} \\ &= (1 - i\sqrt{2})^k \begin{bmatrix} 1 \\ -i\sqrt{2} \end{bmatrix} \\ &= 3^{k/2}(\cos k\theta - i\sin k\theta) \begin{bmatrix} 1 \\ -i\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 3^{k/2}(\cos k\theta + i\sin k\theta) \\ -3^{k/2}(\cos k\theta - i\sin k\theta) \end{bmatrix}, \end{aligned}$$

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where $\theta = \tan^{-1}(\sqrt{2})$. By Corollary 2.2, the system (12) is oscillatory.
If we define $a(k) = \frac{r(k)}{r(k+\ell)}$ and $d(k) = \frac{t(k)}{t(k+\ell)}$ , then $r(k + \ell) = \frac{r(k)}{a(k)}$ and $t(k + \ell) = \frac{t(k)}{d(k)}$
and hence solving the two relations we get

$$r(k) = \frac{r(0)}{\prod_{i=0}^{k-1} a(i)}, \ t(k) = \frac{d(0)}{\prod_{j=0}^{k-1} d(j)},$$

where $r(0)$ and $d(0)$ are non-zero constants if $a(k) \neq 0 \neq d(k)$ for $k \geq k_0 > 0$. From
(4) it follows that

$$r(k + \ell)u(k + \ell) - r(k)u(k) = b(k)r(k + 1)v(k);$$

that is,$\Delta_\ell(r(k)u(k)) = b(k)r(k + \ell)v(k)$.

Consequently, $\sum_{s=0}^{k-1} \Delta_\ell[r(s)u(s)] = \sum_{s=0}^{k-1} b(s)r(s + \ell)v(s)$

that is, $u(k) = \frac{r(0)u(0)}{r(k)} + \frac{1}{r(k)} \sum_{s=0}^{k-1} b(s)r(s + \ell)v(s) = \prod_{i=0}^{k-1} a(i)[u(0) + \sum_{s=0}^{k-1} \frac{b(s)v(s)}{\prod_{i=0}^{s} a(i)}]$

Similarly, $v(k) = \prod_{j=0}^{k-1} d(j)[v(0) + \sum_{s=0}^{k-1} \frac{c(s)u(s)}{\prod_{j=0}^{s} d(j)}]$.

Hence or otherwise the following theorem holds

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