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Hausdorff Property of G^{++-}, G^{+-+} and their Complement Graphs

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Abstract

A simple graph G is said to be Hausdorff graph if for any two vertices u and v of G satisfy at least one of the following conditions: [1] both u and v are isolated [2] either u or v is isolated [3] there exists two non-adjacent edges e_1 and e_2 of G such that e_1 is incident with u and e_2 is incident with v. In this paper, we discuss Hausdorff property on some specific transformation graphs namely $G^{++-}, G^{+-+}, G^{--+}$ and G^{-+-}.

Key words: Hausdorff graph, transformation graph

AMS classification: 05C76, 05C99.

1. Introduction

In [10], Wu Baoyindureng and Meng Jixiang introduced and studied eight types of transformation graph. These transformation graphs have been studied separately by several authors [1, 2, 3, 4, 6, 7].

Let G=(V(G),E(G)) be a simple undirected graph and x,y,z be three variables taking values + or -. The transformation graph G^{xyz} is the graph having $V(G) \cup E(G)$ as the vertex set, and for $\alpha, \beta \in V(G) \cup E(G)$, α and β are adjacent in G^{xyz} if and only if one of the following holds: (i) for $\alpha, \beta \in V(G)$, α and β are adjacent in G if x=+; α and β are not adjacent in G if x=-.(ii) for $\alpha, \beta \in E(G)$, α and β are adjacent in G if y=+; α and β are not adjacent in G if y=-.(iii) for $\alpha \in V(G), \beta \in E(G)$, α and β are incident in G if z=+; α and β are not incident in G if z=-.

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Hausdorff property was introduced and discussed on some derived graphs by Seena and Raji [8, 9]. A graph G is said to be Hausdorff graph if for any two vertices u and v of G satisfy at least one of the following conditions: [1] both u and v are isolated [2] either u or v is isolated [3] there exists two non-adjacent edges e_1 and e_2 of G such that e_1 is incident with u and e_2 is incident with v. Terms not defined are used in the sense of [5]. We use the following theorems for proving results.

Theorem 1.1 [7] Let G be a graph of order $n \geq 6$ and size m. If $m \geq \alpha(G)+1, G^{++-}$ is Hamiltonian.

Theorem 1.2 [8] Any Hamiltonian graph with more than 3 vertices is Hausdorff.

In [3], we obtained necessary and sufficient condition for G^{+++} to be Hausdorff, sufficient condition for G^{---} to be non-Hausdorff and sufficient condition for G^{-++} and G^{+--} to be Hausdorff. In this paper, we obtain results on $G^{++-}, G^{--+}, G^{+-+}$ and G^{-+-}.

2. Result on G^{++-} and G^{--+}

In [7], Lei Yi and Baoyindureng Wu obtained a necessary condition for G^{++-} to be Hamiltonian. In [4], Chandrakala S.B., Manjula K. and Sooryanarayana B. presented degree, eccentricity, diameter, hamiltonicity and vertex independence number of the transformation graph G^{++-}. In this section, we obtain sufficient condition for G^{++-} and G^{--+} to be Hausdorff.

Theorem 2.1 If G is a connected graph of order $n \geq 3$ then G^{++-} is Hausdorff.
Proof: Let u_1 and u_2 be two distinct vertices of G^{++-}.
Case 1: $u_1, u_2 \in V(G)$.
Subcase:(i) u_1 and u_2 are adjacent vertices of G.

Let $e_1 = u_1u_2$. By hypothesis, there exists a vertex u_3 in G which is adjacent to either u_1 or u_2. Let us take $e_2 = u_2u_3$. Then u_1e_2 and u_2u_3 are two non-adjacent edges of G^{++-}.
Subcase:(ii) u_1 and u_2 are non-adjacent vertices of G.

By hypothesis, there exists an edge e_1 incident with u_1 and an edge e_2 incident with u_2 in G. Then u_1e_2 and u_2e_1 are two non-adjacent edges of G^{++-}.
Case 2: $u_1, u_2 \in E(G)$.

Since $u_1 \neq u_2$, there exists two distinct vertices u_3 and u_4 in G such that u_1 is incident with u_3 and u_2 is incident with u_4. Then u_1u_4 and u_2u_3 are two non-adjacent

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edges of G^{++-}.

Case 3: $u_1 \in V(G)$, $u_2 = e_1(say) \in E(G)$.

Subcase:(i) e_1 is incident with u_1 in G.

Let $e_1 = u_1u_3$ be an edge of G. By hypothesis, there exists a vertex u_4 in G which is adjacent to either u_1 or u_3. Let us suppose that u_4 is adjacent to u_3. Then u_1u_3 and e_1u_4 are two non-adjacent edges of G^{++-}.

Subcase:(ii) e_1 is not incident with u_1 in G.

By hypothesis, there exists an edge e_2 incident with u_1.Let $e_2 = u_1u_3$. Suppose e_1 is adjacent to e_2 in G. Then clearly u_1u_3 and e_1e_2 are two non-adjacent edges of G^{++-}. Suppose e_1 is not adjacent to e_2 in G. By hypothesis, there exists an edge e_3 in G which is adjacent to e_1. Then clearly u_1u_3 and e_1e_3 are two non-adjacent edges of G^{++-}. ThusG^{++-} is Hausdorff.

Theorem 2.2 Let G be a disconnected graph of order $n \geq 6$ and size m. If $m \geq \alpha(G) + 1$, G^{++-} is Hausdorff. proof: By Theorem 1.1, G^{++-} is Hamiltonian. Hence by Theorem 1.2, it is Hausdorff.

Theorem 2.3 Let G be a graph of order $n \geq 3$ containing at least one edge. Then G^{--+} is Hausdorff.

Proof: Let $u_1$and u_2 be two distinct vertices of G^{--+}.

Case 1: $u_1, u_2 \in V(G)$.

Subcase:(i) u_1 and u_2 are isolated in G.

Since G contains at least one edge, there exists an edge u_3u_4 in G. Then u_1u_3 and u_2u_4 are two non-adjacent edges of G^{--+}.

Subcase:(ii) u_1 or u_2 is isolated in G.

Suppose u_1 is isolated in G. Since u_2 is not isolated in G, there exists an edge incident with u_2 say $e_1 = u_2u_3$ in G. Then u_1u_3 and u_2e_1 are two non-adjacent edges of G^{--+}.

Subcase:(iii) u_1 and u_2 are adjacent vertices of G.

Let $e_1 = u_1u_2$. Since G is a graph of order $n \geq 3$, there exists a vertex u_3 in G. Suppose u_3 is adjacent to u_1 or u_2 in G. Let us take $e_2 = u_2u_3$. Then u_1e_1 and u_2e_2 are two non-adjacent edges of G^{--+}. Suppose u_3 is not adjacent to u_1 and u_2 in G. Then u_1e_1 and u_2u_3 are two non-adjacent edges of G^{--+}.

Subcase:(iv) u_1 and u_2 are non-adjacent vertices of G.

Since u_1 and u_2 are not isolated in G, there exists an edge e_1 incident with $u_1$and an edge e_2 incident with u_2 in G. Then u_1e_1 and u_2e_2 are two non-adjacent edges of G^{--+}.

Case 2: $u_1, u_2 \in E(G)$.

Since $u_1 \neq u_2$, there exists two distinct vertices u_3 and u_4 in G such that u_1 is

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incident with u_3 and u_2 is incident with u_4. Then u_1u_3 and u_2u_4 are two non-adjacent edges of G^{--+}.

Case 3: $u_1 \in V(G), u_2 = e_1(say) \in E(G)$.

Subcase:(i) e_1 is incident with u_1 in G.

Let $e_1 = u_1u_3$. Since G is a graph of order $n \geq 3$, there exists a vertex u_4 in G. Suppose u_4 is not adjacent to u_1 and u_3 in G. Then u_1u_4 and e_1u_3 are two non-adjacent edges of G^{--+}. Suppose u_4 is adjacent to u_1 in G. Let $e_2 = u_1u_4$. Then u_1e_2 and e_1u_3 are two non-adjacent edges of G^{--+}. Suppose u_4 is adjacent to u_3 and not adjacent to u_1 in G. Then u_1u_4 and e_1u_3 are two non-adjacent edges of G^{--+}.

Subcase:(ii) e_1 is not incident with u_1 in G.

Let $e_1 = u_3u_4$. Suppose u_1 is isolated in G. Then u_1u_4 and e_1u_3 are two non-adjacent edges of G^{--+}. Suppose u_1 is not isolated in G. Then there exists an edge $e_2$incident with u_1 in G. Clearly u_1e_2 and e_1u_3 are two non-adjacent edges of G^{--+}. Therefore, G^{--+} is Hausdorff.

3. Result on G^{+-+} and G^{-+-}

In [6], Lan Xu and Baoyindureng Wu studied connectivity, independence number and hamiltonion of the transformation graph G^{-+-}. Domination number of transformation graphs G^{-+-} and G^{+-+} were studied in [1, 2]. In this section, we obtain necessary and sufficient condition for G^{+-+} and G^{-+-} to be Hausdorff and sufficient condition for G^{-+-} to be non-Hausdorff.

Theorem 3.1 Let G be a connected graph of order $n \geq 4$. Then the transformation graph G^{+-+} is Hausdorff if and only if $G \ncong K_{1,r}$ with $r \geq 3$.

Proof: Assume $G \ncong K_{1,r}$ where $r \geq 3$. Let u_1 and u_2 be two distinct vertices of G^{+-+}.

Case 1: $u_1, u_2 \in V(G)$.

Subcase:(i) u_1 and u_2 are adjacent vertices of G.

Let $e_1 = u_1u_2$. Since G is a connected graph of order $n \geq 4$, there exists a vertex u_3 adjacent to u_1 or u_2 in G. Let u_3 be adjacent to u_1 in G. Then u_1u_3 and u_2e_1 are two non-adjacent edges of G^{+-+}.

Subcase:(ii) u_1 and u_2 are non-adjacent vertices of G.

Since G is connected, there exists an edge e_1 incident with u_1 and an edge e_2 incident with u_2 in G. Then u_1e_1 and u_2e_2 are two non-adjacent edges of G^{+-+}.

Case 2: $u_1, u_2 \in E(G)$.

Since $u_1 \neq u_2$, there exists two distinct vertices u_3 and u_4 in G such that u_1 is incident with u_3 and u_2 is incident with u_4. Then u_1u_3 and u_2u_4 are two non-adjacent edges of G^{+-+}.

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Case 3: $u_1 \in V(G), u_2 = e_1(say) \in E(G)$.
Subcase:(i) e_1 is incident with u_1 in G.

Let $e_1 = u_1u_3$. Since G is a connected graph of order $n \geq 4$,there exists an edge e_2 incident with u_1 or u_3 in G. Let us take $e_2 = u_3u_4$ in G. By assumption, there exists an edge e_3 incident with u_4 in G. Here e_1 and e_3 are not adjacent in G. Then u_1u_3 and e_1e_3 are two non-adjacent edges of G^{+-+}.
Subcase:(ii) e_1 is not incident with u_1 in G.

Let $e_1 = u_3u_4$. Since G is connected, there exists an edge e_2 incident with u_1 in G. Then u_1e_2 and e_1u_4 are two non-adjacent edges of G^{+-+}.Therefore, G^{+-+} is Hausdorff.

Conversely, assume G^{+-+} is Hausdorff. We show that $G \ncong K_{1,r}$ where $r \geq 3$. Suppose $G \cong K_{1,r}$ with $r \geq 3$. Then there exists a vertex u_1 which is adjacent to every other vertices of G. Now consider the vertex $u_2 \neq u_1$ and an edge $e_1 = u_1u_2$ in G. Then u_2u_1 and u_2e_1 are the only edges incident with u_2 and e_1u_2 and e_1u_1 are the only edges incident with e_1 in G^{+-+}. Therefore for the two vertices u_2 and e_1 in G^{+-+}, Hausdorff property is not true. This is a contradiction to our assumption.

Hence $G \ncong K_{1,r}$ where $r \geq 3$.

Theorem 3.2 Let G be a graph. If $G \cong K_{1,r} \cup K_1, r \geq 1$ or $G \cong K_{1,r} + e, r > 1$ then G^{-+-} is not Hausdorff.
Proof: Suppose $G \cong K_{1,r} \cup K_1$ with $r \geq 1$. Then G consists of an isolated vertex $u_1$and a vertex u_2 which is adjacent to every other vertices of G other than u_1. Hence by the definition of the transformation graph G^{-+-}, u_2 is adjacent to u_1 in G^{-+-} and no other vertices of G^{-+-} is adjacent to u_2. So u_2 is a pendant vertex in G^{-+-}. Therefore G^{-+-} is not Hausdorff.
Suppose $G \cong K_{1,r+1} + e(r > 1)$. Then G consists of a vertex u which is adjacent to every other vertices of G and e is an edge of G not incident with u. Hence by the definition of the transformation graph G^{-+-}, u is adjacent to e in G^{-+-} and no other vertices of G^{-+-} is adjacent to u. So u is a pendant vertex in G^{-+-}. Therefore G^{-+-} is not Hausdorff.

Theorem 3.3 Let G be a connected graph of order $n \geq 4$. Then the transformation graph G^{-+-} is Hausdorff if and only if $G \cong K_{1,r} + e$ where $r \geq 3$.
Proof: Assume $G \ncong K_{1,r} + e$ where $r \geq 3$. We shall show that G^{-+-} is Hausdorff.
Let u_1 and u_2 be two distinct vertices of G^{-+-}.
Case 1: $u_1, u_2 \in V(G)$.
Subcase:(i) u_1 and u_2 are adjacent vertices of G. Since G is a connected graph of order $n \geq 4$, there exists an edge e_1 incident with u_1 or u_2. Let us assume e_1 is incident with u_2 in G. Suppose $G \cong K_{1,r}$. Then u_2 is isolated in G^{-+-}. Suppose

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$G \ncong K_{1,r}$. There exists an edge e_2 not incident with u_2 in G. Then u_1e_1 and u_2e_2 are two non-adjacent edges of G^{-+-}.
Subcase:(ii) u_1 and u_2 are non-adjacent vertices of G. Since G is connected, there exists an edge e_1 incident with u_1 and an edge e_2 incident with u_2 in G. Then u_1e_2 and u_2e_1 are two non-adjacent edges of G^{-+-}.
Case 2: $u_1, u_2 \in E(G)$.

Since $u_1 \neq u_2$, there exists two distinct vertices u_3 and u_4 in G such that u_1 is incident with u_3 and u_2 is incident with u_4. Then u_1u_4 and u_2u_3 are two non-adjacent edges of G^{-+-}.
Case 3: $u_1 \in V(G), u_2 = e_1 \ \ (say) \in E(G)$.
Subcase:(i) e_1 is incident with u_1 in G.

Let $e_1 = u_1u_3$. Since G is a connected graph of order $n \geq 4$, there exists at least two edges e_2 and e_3 in G. Suppose $e_2 = u_3u_4$ and e_3 is any edge in G. Then u_1e_2 and e_1u_4 are two non-adjacent edges of G^{-+-}. Suppose $e_2 = u_1u_4$ and $e_3 = u_4u_5$ in G. Then u_1e_3 and e_1e_2 are two non-adjacent edges of G^{-+-}. Suppose e_2 and e_3 are incident with $u_1$and there is no edge without incident with u_1 in G.
Then $u_1$is isolated in G^{-+-}.
Subcase:(ii) e_1 is not incident with u_1 in G.

Let $e_2 = u_1u_3$. Since G is a connected graph of order $n \geq 4$, there exists an edge e_3 in G. Suppose $e_1 = u_3u_4$ and e_3 is any edge in G. Since G is a connected graph of order $n \geq 4$, there exists at least one edge (may be e_3) not incident with u_1 or there exists at least one vertex (may be u_4) not adjacent to u_1 in G. Let it be u_5. Then u_1u_5 and e_1e_2 are two non-adjacent edges of G^{-+-}.
Suppose $e_1 = u_4u_5$ and $e_3 = u_1u_4$ or $e_1 = u_4u_5$ and $e_3 = u_3u_4$ in G. By assumption, there exists at least one edge different from e_1 and e_3 not incident with u_1 or there exists at least one vertex (may be u_5) not adjacent to u_1 in G. Let it be u_6. Then u_1u_6 and e_1e_3 are two non-adjacent edges of G^{-+-}.
Therefore, G^{-+-} is Hausdorff.
Conversely, assume G^{-+-} is Hausdorff. By theorem 3.2, $G \ncong K_{1,r} + e$ where $r \geq 3$.

Theorem 3.4 Let G be a disconnected graph of order $n \geq 4$.Then the transformation graph G^{-+-} is Hausdorff if and only if $G \ncong K_{1,r} \cup K_1$ where $r \geq 2$.
Proof: Assume $G \ncong K_{1,r} \cup K_1$ where $r \geq 2$. We shall show that G^{-+-} is Hausdorff. Let u_1 and u_2 be two distinct vertices of G^{-+-}.
Case 1: $u_1, u_2 \in V(G)$.
Subcase:(i) u_1 and u_2 are isolated in G.

Since G is a graph of order $n \geq 4$, there exists at least two vertices u_3 and u_4 different from u_1 and u_2 in G. Then u_1u_3 and u_2u_4 are two non-adjacent edges of G^{-+-}.

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Subcase:(ii) u_1 or u_2 is isolated in G.

Suppose u_1 is isolated in G. Since u_2 is not isolated in G, there exists an edge e_1 incident with u_2 in G. By assumption, there exists at least one edge not incident with u_2 or there exists at least one vertex different from u_1 not adjacent to u_2 in G. Let it be u_3. Then u_1e_1 and u_2u_3 are two non-adjacent edges of G^{-+-}.

Subcase:(iii) u_1 and u_2 are adjacent vertices of G.

Since G is disconnected, there exists at least one vertex u_3 not adjacent to u_1 and u_2 in G. By assumption, there exists at least one edge not incident with u_2 or there exists at least one vertex different from u_3 not adjacent to u_2 in G. Let it be u_4. Then u_1u_3 and u_2u_4 are two non-adjacent edges of G^{-+-}.

Subcase:(iv) u_1 and u_2 are non-adjacent vertices of G.

Since u_1 and u_2 are not isolated in G, there exists two distinct edges e_1 and e_2 in G such that e_1 is incident with u_1 and e_2 is incident with u_2. Then $u_1e_2$and u_2e_1 are two non-adjacent edges of G^{-+-}.

Case 2: $u_1, u_2 \in E(G)$.

Since $u_1 \neq u_2$, there exists two distinct vertices u_3 and u_4 in G such that u_1 is incident with u_3 and u_2 is incident with u_4. Then u_1u_4 and u_2u_3 are two non-adjacent edges of G^{-+-}.

Case 3: $u_1 \in V(G), u_2 = e_1 \ \ (say) \in E(G)$.

Subcase:(i) e_1 is incident with u_1 in G.

Since G is disconnected, there exists at least one vertex u_3 not incident with e_1 and not adjacent to u_1 in G. By assumption, there exists at least one edge not incident with u_1 or there exists at least one vertex different from u_3 not adjacent to u_1 in G. Let it be u_4. Then u_1u_4 and e_1u_3 are two non-adjacent edges of G^{-+-}.

Subcase:(ii) e_1 is not incident with u_1 in G.

Suppose u_1 and e_1 belong to different components of G. Let $e_1 = u_3u_4$. By assumption, there exists at least one edge adjacent to e_1 or there exists at least one vertex different from u_1 is not incident with e_1 in G. Let it be u_5. Then u_1u_3 and e_1u_5 are two non-adjacent edges of G^{-+-}.

Suppose u_1 and e_1 belong to same component of G. Then there exists an edge e_2 adjacent to e_1 in G. Since G is disconnected, there exists at least one vertex u_4 not incident with e_1 and not adjacent to u_1 in G. Then u_1u_4 and e_1e_2 are two non-adjacent edges of G^{-+-}.

Therefore, G^{-+-} is Hausdorff.

Conversely, assume G^{-+-} is Hausdorff. By theorem 3.2 $G \ncong K_{1,r} \cup K_1$ where $r \geq 2$.

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