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A Short Note on Pairwise Fuzzy Irresolvable Spaces and Pairwise Fuzzy Open Hereditarily Irresolvable Spaces

G. Thangaraj[1*], P. Vivakanandan[2**]

[1] Department of Mathematics, Thiruvalluvar University, Vellore-632115, Tamil Nadu, INDIA.

[2]Department of Mathematics, Jawahar Science College, Neyveli-607803, Tamil Nadu, INDIA.

Abstract

The main aim of this paper is the several characterizations of pairwise fuzzy irresolvable spaces and pairwise fuzzy open hereditarily irresolvable spaces are established.

Key words: Pairwise fuzzy semi-open set, pairwise fuzzy open set, pairwise fuzzy clopen set, pairwise fuzzy dense set, pairwise fuzzy nowhere dense set, pairwise fuzzy somewhere dense set, pairwise fuzzy regular open set, pairwise fuzzy faintly open set, pairwise fuzzy preopen set, pairwise fuzzy resolvable space, pairwise fuzzy irresolvable space and pairwise fuzzy submaximal space.

AMS classification: 54 A 40, 03 E 72

1.Introduction

The fuzzy sets were introduced by L.A.Zadeh [23] in 1965 and further the concept of fuzzy topological spaces was introduced by C.L.Chang [3] which is the generalization of topological spaces in 1968. A.Kandil [7] introduced the concept of fuzzy bitopological spaces in 1989 which is the generalization of fuzzy bitopological spaces and since then many concepts in classical topology have been extended to fuzzy bitopological spaces. K.K.Azad [1], R.Lowen [11], A.K.Katsaras [10], C.K.Wong [22], R.H.Warren [21] and many others have contributed a lot to the field of fuzzy topology. In 1969, A.G.El'Kin [5] introduced and studied open hereditarily irresolvable spaces in classical topology. The study of El'Kin interrelated open hereditarily irresolvable spaces with a collection of pre-open sets. C.Chattopadhyay and U.K.Roy [4] studied

*g.thangaraj@rediffmail.com, **vivekjsc@gmail.com

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open hereditarily irresolvable spaces extensively in classical topology. This concept in fuzzy setting was introduced by G.Thangaraj and G.Balasubramanian [15]. The concept of pairwise fuzzy open hereditarily irresolvability of fuzzy bitopological spaces was introduced by G.Thangaraj [14]. In this paper, several characterizations of pairwise fuzzy irresolvable spaces and pairwise fuzzy open hereditarily irresolvable spaces are established.

2.Preliminaries

Now we give some basic notions and results used in the sequel. In this work by (X, T) or simply by X, we will denote a fuzzy topological space due to Chang (1968). By a fuzzy bitopological space (Kandil, 1989) we mean an ordered triple (X, T_1, T_2), where T_1 and T_2 are two fuzzy topologies on a non-empty set X. Throughout this paper, the indices i and j take values in $\{1, 2\}$ and $i \neq j$.

Definition 2.1 [3] Let (X, T) be any fuzzy topological space and λ be any fuzzy set in (X, T). The closure and interior of a fuzzy set λ in a fuzzy topological space (X, T) are respectively denoted as $cl(\lambda)$ and $int(\lambda)$ are defined as

(1) $cl(\lambda) = \wedge\{\mu \mid \lambda \leq \mu, 1 - \mu \in T\}$ and
(2) $int(\lambda) = \vee\{\mu \mid \mu \leq \lambda, \mu \in T\}$.

Lemma 2.2 [1] For a fuzzy set λ of a fuzzy space X,

(a) $1 - cl(\lambda) = int(1 - \lambda)$ and
(b) $1 - int(\lambda) = cl(1 - \lambda)$.

Lemma 2.3 [1] For a family $\mathcal{A} = \{\lambda_a\}$ of fuzzy sets of a fuzzy space X. Then, $\vee\, cl(\lambda_\alpha) \leq cl(\vee\lambda_\alpha)$. In case $\mathcal{A}$ is a finite set, $\vee\, cl(\lambda_\alpha) = cl(\vee\lambda_\alpha)$. Also $\vee\, int\,(\lambda_\alpha) \leq int(\vee\lambda_\alpha)$.

Theorem 2.4 [21] Let X be a fuzzy topological space and λ, μ be fuzzy sets in X. Then we have

(1). λ is fuzzy closed (resp., fuzzy open) $\Leftrightarrow cl(\lambda) = \lambda$ (resp., $int(\lambda) = \lambda$);
(2). $\lambda \leq \mu \Rightarrow cl(\lambda) \leq cl(\mu) \quad (int(\lambda) \leq int(\mu))$;
(3). $cl\ cl(\lambda) = cl(\lambda) \quad (int\ int(\lambda) = int(\lambda))$;
(4). $cl(\lambda) \vee cl(\mu) = cl(\lambda \vee \mu)$;
(5). $cl(\lambda) \wedge cl(\mu) \geq cl(\lambda \wedge \mu)$;

(6). $int(\lambda) \vee int(\mu) \leq int(\lambda \vee \mu)$;
(7). $int(\lambda) \wedge int(\mu) = int(\lambda \wedge \mu)$;

Definition 2.5 [23] A fuzzy set λ in a set X is a function from X to $[0, 1]$, that is., $\lambda : X \to [0, 1]$.

Definition 2.6 [16] A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy open set if $\lambda \in T_i,\ (i = 1, 2)$.

Definition 2.7 [16] A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy closed set if $1 - \lambda \in T_i,\ (i = 1, 2)$.

Definition 2.8 [13] A fuzzy set A of a fuzzy bitopological space (X, τ_1, τ_2) is called

(a) (i, j)-fuzzy semi-open if there exists a τ_i-fuzzy open set U such that $U \leq A \leq \tau_j - Cl(U)$.
(b) (i, j)-fuzzy semi-closed if there exists a τ_i-fuzzy closed set F such that $\tau_j - int(F) \leq A \leq F$.

Definition 2.9 [14] A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy dense set if $cl_{T_1} cl_{T_2}(\lambda) = 1$ and $cl_{T_2} cl_{T_1}(\lambda) = 1$.

Definition 2.10 [14] A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy nowhere dense set if $int_{T_1} cl_{T_2}(\lambda) = 0$ and $int_{T_2} cl_{T_1}(\lambda) = 0$.

Definition 2.11 [16] A non-zero fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy somewhere dense set if $int_{T_1} cl_{T_2}(\lambda) \neq 0$ and $int_{T_2} cl_{T_1}(\lambda) \neq 0$.

Definition 2.12 [9] Two fuzzy sets A and B in a set X are said to be disjoint if $A \bigodot B = \emptyset$, where $(A \bigodot B)(x) = max\{0, A(x) + B(x) - 1\}$ for all $x \in X$. It is clear that $A \bigodot B = \emptyset$ if and only if $A \subset B^C$.

Definition 2.13 A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a

(a.) pairwise fuzzy regular open set in (X, T_1, T_2) if $int_{T_1} cl_{T_2}(\lambda) = \lambda = int_{T_2} cl_{T_1}(\lambda)$ [2].

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(b.) pairwise fuzzy regular closed set in (X, T_1, T_2) if $cl_{T_1}int_{T_2}(\lambda) = \lambda = cl_{T_2}int_{T_1}(\lambda)$ [2].

(c.) pairwise fuzzy pre-open set in (X, T_1, T_2) if $\lambda \leq int_{T_i}cl_{T_j}(\lambda)$, $(i \neq j$ and $i, j = 1, 2)$ [12].

(d.) pairwise fuzzy pre-closed set in (X, T_1, T_2) if $cl_{T_i}int_{T_j}(\lambda) \leq \lambda$, $(i \neq j$ and $i, j = 1, 2)$ [12].

Definition 2.14 [19] A fuzzy bitopological space (X, T_1, T_2) is said to be a pairwise fuzzy submaximal space if for each fuzzy set λ in (X, T_1, T_2) such that $cl_{T_i}(\lambda) = 1$, $(i = 1, 2)$, λ is a pairwise fuzzy open set in (X, T_1, T_2).

Theorem 2.15 [20] If a fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy resolvable space, then there exists a fuzzy set λ defined on X such that

(i) $cl_{T_1}cl_{T_2}(\lambda) = 1 = cl_{T_2}cl_{T_1}(\lambda)$ and $int_{T_1}int_{T_2}(\lambda) = 0 = int_{T_2}int_{T_1}(\lambda)$ in (X, T_1, T_2).

(ii) $int_{T_1}int_{T_2}(\lambda) = 0 = int_{T_2}int_{T_1}(\lambda)$ and $int_{T_1}int_{T_2}(1-\lambda) = 0 = int_{T_2}int_{T_1}(1-\lambda)$ in (X, T_1, T_2).

3.Pairwise fuzzy irresolvable spaces

Definition 3.1 A fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy resolvable space if there exists a pairwise fuzzy dense set λ in (X, T_1, T_2) such that $1 - \lambda$ is also a pairwise fuzzy dense set in (X, T_1, T_2).

That is, (X, T_1, T_2) is called a pairwise fuzzy resolvable space if there exists a fuzzy set λ defined on X such that $cl_{T_1}cl_{T_2}(\lambda) = 1 = cl_{T_2}cl_{T_1}(\lambda)$ and $cl_{T_1}cl_{T_2}(1-\lambda) = 1 = cl_{T_2}cl_{T_1}(1-\lambda)$ in (X, T_1, T_2).

Definition 3.2 [14] A fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy irresolvable space if for each pairwise fuzzy dense set λ in (X, T_1, T_2), $cl_{T_1}cl_{T_2}(1-\lambda) \neq 1$ and $cl_{T_2}cl_{T_1}(1-\lambda) \neq 1$ in (X, T_1, T_2).

Example 3.3 Let $X = \{a, b, c\}$. The fuzzy sets λ, μ, δ are defined on X as follows:

$\lambda : X \to [0, 1]$ is defined as $\lambda(a) = 0.25;\quad \lambda(b) = 0;\quad \lambda(c) = 0$

$\mu : X \to [0, 1]$ is defined as $\mu(a) = 0.75;\quad \mu(b) = 0.5;\quad \mu(c) = 0$

$\delta : X \to [0, 1]$ is defined as $\delta(a) = 0.8;\quad \delta(b) = 0;\quad \delta(c) = 1.$

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Clearly, $T_1 = \{0, \lambda, \mu, 1\}$ and $T_2 = \{0, \lambda, \delta, 1\}$ are fuzzy topologies on X. On computation, one can see that the only pairwise fuzzy dense set in (X, T_1, T_2) is δ and $cl_{T_1}cl_{T_2}(1-\delta) = 1-\lambda \neq 1$ and $cl_{T_2}cl_{T_1}(1-\delta) = 1-\lambda \neq 1$. Hence, for the pairwise fuzzy dense set δ in (X, T_1, T_2), $1-\delta$ is not a pairwise fuzzy dense set in (X, T_1, T_2). Therefore, (X, T_1, T_2) is a pairwise fuzzy irresolvable space.

4.Characterizations of pairwise fuzzy irresolvable spaces

Proposition 4.1 If a fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy irresolvable space, then for each pairwise fuzzy dense set λ in (X, T_1, T_2), $cl_{T_1}int_{T_2}(\lambda) \neq 0$ and $cl_{T_2}int_{T_1}(\lambda) \neq 0$ in (X, T_1, T_2).

Proof: Let λ be a pairwise fuzzy dense set in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy irresolvable space, for the pairwise fuzzy dense set λ in (X, T_1, T_2), $cl_{T_1}cl_{T_2}(1-\lambda) \neq 1$ and $cl_{T_2}cl_{T_1}(1-\lambda) \neq 1$ in (X, T_1, T_2). Now $int_{T_1}cl_{T_2}(1-\lambda) \leq cl_{T_1}cl_{T_2}(1-\lambda)$ and $int_{T_2}cl_{T_1}(1-\lambda) \leq cl_{T_2}cl_{T_1}(1-\lambda)$ in (X, T_1, T_2). Then, $int_{T_1}cl_{T_2}(1-\lambda) \neq 1$ and $int_{T_2}cl_{T_1}(1-\lambda) \neq 1$ in (X, T_1, T_2) and hence $1 - cl_{T_1}int_{T_2}(\lambda) \neq 1$ and $1 - cl_{T_2}int_{T_1}(\lambda) \neq 1$ in (X, T_1, T_2). Thus, $cl_{T_1}int_{T_2}(\lambda) \neq 0$ and $cl_{T_2}int_{T_1}(\lambda) \neq 0$ in (X, T_1, T_2).

Proposition 4.2 If $int_{T_1}int_{T_2}(\lambda) = 0 = int_{T_2}int_{T_1}(\lambda)$, for a fuzzy set λ in a pairwise fuzzy irresolvable space (X, T_1, T_2), then $cl_{T_1}cl_{T_2}(\lambda) \neq 1$ and $cl_{T_2}cl_{T_1}(\lambda) \neq 1$, in (X, T_1, T_2).

Proof: Suppose that $int_{T_1}int_{T_2}(\lambda) = 0 = int_{T_2}int_{T_1}(\lambda)$, in (X, T_1, T_2). Then $1 - int_{T_1}int_{T_2}(\lambda) = 1 - 0 = 1$ and $1 - int_{T_2}int_{T_1}(\lambda) = 1 - 0 = 1$ and hence $cl_{T_1}cl_{T_2}(1-\lambda) = 1$ and $cl_{T_2}cl_{T_1}(1-\lambda) = 1$, in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy irresolvable space, for the pairwise fuzzy dense set $1-\lambda$ in (X, T_1, T_2), $cl_{T_1}cl_{T_2}(1-(1-\lambda)) \neq 1$ and $cl_{T_2}cl_{T_1}(1-(1-\lambda)) \neq 1$, in (X, T_1, T_2). That is, $cl_{T_1}cl_{T_2}(\lambda) \neq 1$ and $cl_{T_2}cl_{T_1}(\lambda) \neq 1$, in (X, T_1, T_2).

Proposition 4.3 If λ is a pairwise fuzzy nowhere dense set in a pairwise fuzzy irresolvable space (X, T_1, T_2), then λ is not a pairwise fuzzy dense set in (X, T_1, T_2).

Proof: Let λ be a pairwise fuzzy nowhere dense set in (X, T_1, T_2). Then $int_{T_1}cl_{T_2}(\lambda) = 0 = int_{T_2}cl_{T_1}(\lambda)$, in (X, T_1, T_2). Now $int_{T_1}int_{T_2}(\lambda) \leq int_{T_1}cl_{T_2}(\lambda)$ and $int_{T_2}int_{T_1}(\lambda) \leq int_{T_2}cl_{T_1}(\lambda)$, in (X, T_1, T_2), implies that $int_{T_1}int_{T_2}(\lambda) \leq 0$ and $int_{T_2}int_{T_1}(\lambda) \leq 0$, in (X, T_1, T_2). That is, $int_{T_1}int_{T_2}(\lambda) = 0$ and $int_{T_2}int_{T_1}(\lambda) =$

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0, in (X, T_1, T_2). Then, since (X, T_1, T_2) is a pairwise fuzzy irresolvable space and $int_{T_1} int_{T_2}(\lambda) = 0$ and $int_{T_2} int_{T_1}(\lambda) = 0$ in (X, T_1, T_2), by proposition 4.2, $cl_{T_1} cl_{T_2}(\lambda) \neq 1$ and $cl_{T_2} cl_{T_1}(\lambda) \neq 1$, in (X, T_1, T_2). Hence, λ is not a pairwise fuzzy dense set in (X, T_1, T_2).

Proposition 4.4 If a fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy irresolvable space, then $\lambda_1 \wedge \lambda_2 \neq 0$, for any two pairwise fuzzy dense sets λ_1 and λ_2 in (X, T_1, T_2).

Proof: Let (X, T_1, T_2) be a pairwise fuzzy irresolvable space and λ_1 and λ_2 be any two pairwise fuzzy dense sets in (X, T_1, T_2). Then $cl_{T_1} cl_{T_2}(\lambda_1) = 1 = cl_{T_2} cl_{T_1}(\lambda_1)$ and $cl_{T_1} cl_{T_2}(\lambda_2) = 1 = cl_{T_2} cl_{T_1}(\lambda_2)$ It has to be proved that $\lambda_1 \wedge \lambda_2 \neq 0$ in (X, T_1, T_2). Assume the contrary. Suppose that $\lambda_1 \wedge \lambda_2 = 0$ in (X, T_1, T_2). Then λ_1 and λ_2 are any two disjoint fuzzy sets in (X, T_1, T_2) and hence by definition 2.12, $\lambda_1 \leq (1 - \lambda_2)$ in (X, T_1, T_2). Then $cl_{T_1} cl_{T_2}(\lambda_1) \leq cl_{T_1} cl_{T_2}(1 - \lambda_2)$ and $cl_{T_2} cl_{T_1}(\lambda_1) \leq cl_{T_2} cl_{T_1}(1 - \lambda_2)$ in (X, T_1, T_2) and hence $1 \leq cl_{T_1} cl_{T_2}(1 - \lambda_2)$ and $1 \leq cl_{T_2} cl_{T_1}(1 - \lambda_2)$ in (X, T_1, T_2). That is, $cl_{T_1} cl_{T_2}(1 - \lambda_2) = 1$ and $cl_{T_2} cl_{T_1}(1 - \lambda_2) = 1$ in (X, T_1, T_2). Thus, there exists a pairwise fuzzy dense set λ_2 in (X, T_1, T_2) such that $cl_{T_1} cl_{T_2}(1 - \lambda_2) = 1 = cl_{T_2} cl_{T_1}(1 - \lambda_2)$. This means that (X, T_1, T_2) is a pairwise fuzzy resolvable space, a contradiction to (X, T_1, T_2) being a pairwise fuzzy irresolvable space. Hence, the assumption that $\lambda_1 \wedge \lambda_2 = 0$ in (X, T_1, T_2) does not hold. Therefore, $\lambda_1 \wedge \lambda_2 \neq 0$, for any two pairwise fuzzy dense sets λ_1 and λ_2 in a pairwise fuzzy irresolvable space (X, T_1, T_2).

Definition 4.5 A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy faintly open set if either $\lambda = 0$ or $int_{T_1} int_{T_2}(\lambda) \neq 0$ and $int_{T_2} int_{T_1}(\lambda) \neq 0$, in (X, T_1, T_2).

Proposition 4.6 A fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy irresolvable space if and only if each pairwise fuzzy dense set is a pairwise fuzzy faintly open set in (X, T_1, T_2).

Proof: Let (X, T_1, T_2) be a pairwise fuzzy irresolvable space and λ be a pairwise fuzzy dense set in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy irresolvable space, for the pairwise fuzzy dense set λ in (X, T_1, T_2), $cl_{T_1} cl_{T_2}(1 - \lambda) \neq 1$ and $cl_{T_2} cl_{T_1}(1 - \lambda) \neq 1$ in (X, T_1, T_2). Then $1 - int_{T_1} int_{T_2}(\lambda) \neq 1$ and $1 - int_{T_2} int_{T_1}(\lambda) \neq 1$ in (X, T_1, T_2) and hence $int_{T_1} int_{T_2}(\lambda) \neq 0$ and $int_{T_2} int_{T_1}(\lambda) \neq 0$ in (X, T_1, T_2). Therefore λ is a pairwise fuzzy faintly open set in (X, T_1, T_2).

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Conversely, let each pairwise fuzzy dense set λ be a pairwise fuzzy faintly open set in (X, T_1, T_2). Then $\lambda \neq 0$. [otherwise $\lambda = 0$ will imply that $cl_{T_1}cl_{T_2}(\lambda) = cl_{T_1}cl_{T_2}(0) = 0 \neq 1$ and $cl_{T_2}cl_{T_1}(\lambda) = cl_{T_2}cl_{T_1}(0) = 0 \neq 1$, a contradiction to λ being a pairwise fuzzy dense set in (X, T_1, T_2).] It has to be proved that (X, T_1, T_2) is a pairwise fuzzy irresolvable space. Assume the contrary. Suppose that (X, T_1, T_2) is a pairwise fuzzy resolvable space. Then, there exists a pairwise fuzzy dense set μ in (X, T_1, T_2) such that $cl_{T_1}cl_{T_2}(1 - \mu) = 1$ and $cl_{T_2}cl_{T_1}(1 - \mu) = 1$ in (X, T_1, T_2) and hence $1 - int_{T_1}int_{T_2}(\mu) = 1$ and $1 - int_{T_2}int_{T_1}(\mu) = 1$ in (X, T_1, T_2). Then $int_{T_1}int_{T_2}(\mu) = 0$ and $int_{T_2}int_{T_1}(\mu) = 0$ in (X, T_1, T_2), a contraction to the hypothesis that each pairwise fuzzy dense set is a pairwise fuzzy faintly open set in (X, T_1, T_2). Hence, the assumption that (X, T_1, T_2) is a pairwise fuzzy resolvable space, does not hold and therefore (X, T_1, T_2) is a pairwise fuzzy irresolvable space,

The following proposition gives a condition, in terms of pairwise fuzzy dense set, for a fuzzy bitopological space to be a pairwise fuzzy irresolvable space.

Proposition 4.7 If $int_{T_1}int_{T_2}(\lambda) \neq 0$ and $int_{T_2}int_{T_1}(\lambda) \neq 0$, for each pairwise fuzzy dense set λ in a fuzzy bitopological space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy irresolvable space.

Proof: Let λ be a pairwise fuzzy dense set in (X, T_1, T_2) such that $int_{T_1}int_{T_2}(\lambda) \neq 0$ and $int_{T_2}int_{T_1}(\lambda) \neq 0$. Now $cl_{T_1}cl_{T_2}(1 - \lambda) = 1 - int_{T_1}int_{T_2}(\lambda) \neq 1$ and $cl_{T_2}cl_{T_1}(1 - \lambda) = 1 - int_{T_2}int_{T_1}(\lambda) \neq 1$. Hence, for the pairwise fuzzy dense set λ in (X, T_1, T_2), $cl_{T_1}cl_{T_2}(1 - \lambda) \neq 1$ and $cl_{T_2}cl_{T_1}(1 - \lambda) \neq 1$ in (X, T_1, T_2) and this is true for all pairwise fuzzy dense sets in (X, T_1, T_2) and hence for each pairwise fuzzy dense set in (X, T_1, T_2), $cl_{T_1}cl_{T_2}(1 - \lambda) \neq 1$ and $cl_{T_2}cl_{T_1}(1 - \lambda) \neq 1$. Therefore, (X, T_1, T_2) is a pairwise fuzzy irresolvable space.

The following proposition gives a condition for a fuzzy bitopological space to be a pairwise fuzzy irresolvable space.

Proposition 4.8 If $int_{T_1}cl_{T_2}(\lambda) \leq cl_{T_1}int_{T_2}(\lambda)$ and $int_{T_2}cl_{T_1}(\lambda) \leq cl_{T_2}int_{T_1}(\lambda)$, for each fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy irresolvable space.

Proof: Let λ be a fuzzy set in (X, T_1, T_2) such that $int_{T_1}cl_{T_2}(\lambda) \leq cl_{T_1}int_{T_2}(\lambda)$ and $int_{T_2}cl_{T_1}(\lambda) \leq cl_{T_2}int_{T_1}(\lambda)$. Now $int_{T_1}cl_{T_2}(\lambda) \leq cl_{T_1}int_{T_2}(\lambda)$, implies that $int_{T_1}cl_{T_2}(\lambda) - cl_{T_1}int_{T_2}(\lambda) \leq 0$. That is, $int_{T_1}cl_{T_2}(\lambda) - cl_{T_1}int_{T_2}(\lambda) = 0$ and hence

$int_{T_1}cl_{T_2}(\lambda)\wedge[1-cl_{T_1}int_{T_2}(\lambda)]=0$, in (X,T_1,T_2). Then, $int_{T_1}cl_{T_2}(\lambda)\wedge int_{T_1}cl_{T_2}(1-\lambda)$ $=0$ in (X,T_1,T_2). Now $int_{T_1}[cl_{T_2}(\lambda)\wedge cl_{T_2}(1-\lambda)]=int_{T_1}cl_{T_2}(\lambda)\wedge int_{T_1}cl_{T_2}(1-\lambda)$ $=0$, in (X,T_1,T_2). This implies that $int_{T_1}[cl_{T_2}(\lambda)\wedge cl_{T_2}(1-\lambda)]=0$. Then either $cl_{T_2}(\lambda)\wedge cl_{T_2}(1-\lambda)=0$ or $cl_{T_2}(\lambda)\wedge cl_{T_2}(1-\lambda)\neq 0$ in (X,T_1,T_2).

Suppose that $cl_{T_2}(\lambda)\wedge cl_{T_2}(1-\lambda)=0$ in (X,T_1,T_2). Then $cl_{T_2}(\lambda)\wedge[1-int_{T_2}(\lambda)]=0$, implies that $cl_{T_2}(\lambda)\wedge[int_{T_2}(\lambda)]'=0$ and hence $cl_{T_2}(\lambda)-int_{T_2}(\lambda)=0$. Then $cl_{T_2}(\lambda)=$ $int_{T_2}(\lambda)$, a contradiction. [This is applicable only in the case of pairwise fuzzy clopen sets for which $cl_{T_2}(\lambda)=\lambda=int_{T_2}(\lambda)$]. Hence it must be that $cl_{T_2}(\lambda)\wedge cl_{T_2}(1-\lambda)\neq 0$, in (X,T_1,T_2). Now by theorem 2.4, $cl_{T_1}[cl_{T_2}(\lambda)\wedge cl_{T_2}(1-\lambda)]\leq cl_{T_1}cl_{T_2}(\lambda)\wedge cl_{T_1}cl_{T_2}(1-\lambda)$. Then, $cl_{T_1}cl_{T_2}(\lambda)\wedge cl_{T_1}cl_{T_2}(1-\lambda)\neq 0$.......$(A)$ in (X,T_1,T_2).

Also, $int_{T_2}cl_{T_1}(\lambda)\leq cl_{T_2}int_{T_1}(\lambda)$, implies that $int_{T_2}cl_{T_1}(\lambda)-cl_{T_2}int_{T_1}(\lambda)\leq 0$. That is, $int_{T_2}cl_{T_1}(\lambda)-cl_{T_2}int_{T_1}(\lambda)=0$ and hence $int_{T_2}cl_{T_1}(\lambda)\wedge[1-cl_{T_2}int_{T_1}(\lambda)]=0$, in (X,T_1,T_2). Then, $int_{T_2}cl_{T_1}(\lambda)\wedge int_{T_2}cl_{T_1}(1-\lambda)=0$ in (X,T_1,T_2). Now $int_{T_2}[cl_{T_1}(\lambda)\wedge cl_{T_1}(1-\lambda)]=int_{T_2}cl_{T_1}(\lambda)\wedge int_{T_2}cl_{T_1}(1-\lambda)=0$, in (X,T_1,T_2). This implies that $int_{T_2}[cl_{T_1}(\lambda)\wedge cl_{T_1}(1-\lambda)]=0$. Then either $cl_{T_1}(\lambda)\wedge cl_{T_1}(1-\lambda)=0$ or $cl_{T_1}(\lambda)\wedge cl_{T_1}(1-\lambda)\neq 0$ in (X,T_1,T_2).

Suppose that $cl_{T_1}(\lambda)\wedge cl_{T_1}(1-\lambda)=0$ in (X,T_1,T_2). Then $cl_{T_1}(\lambda)\wedge[1-int_{T_1}(\lambda)]=0$, implies that $cl_{T_1}(\lambda)\wedge[int_{T_1}(\lambda)]'=0$ and hence $cl_{T_1}(\lambda)-int_{T_1}(\lambda)=0$. Then $cl_{T_1}(\lambda)=$ $int_{T_1}(\lambda)$, a contradiction. [This is applicable only in the case of pairwise fuzzy clopen sets for which $cl_{T_1}(\lambda)=\lambda=int_{T_1}(\lambda)$]. Hence it must be that $cl_{T_1}(\lambda)\wedge cl_{T_1}(1-\lambda)\neq 0$, in (X,T_1,T_2). Now by theorem 2.4, $cl_{T_2}[cl_{T_1}(\lambda)\wedge cl_{T_1}(1-\lambda)]\leq cl_{T_2}cl_{T_1}(\lambda)\wedge cl_{T_2}cl_{T_1}(1-\lambda)$. Then, $cl_{T_2}cl_{T_1}(\lambda)\wedge cl_{T_2}cl_{T_1}(1-\lambda)\neq 0$.......$(B)$ in (X,T_1,T_2).

If λ is a pairwise fuzzy dense set in (X,T_1,T_2), then $cl_{T_1}cl_{T_2}(\lambda)=1=cl_{T_2}cl_{T_1}(\lambda)$ in (X,T_1,T_2). Then from (A) and (B), $1\wedge cl_{T_1}cl_{T_2}(1-\lambda)\neq 0$ and $1\wedge cl_{T_2}cl_{T_1}(1-\lambda)\neq 0$. Thus, $cl_{T_1}cl_{T_2}(1-\lambda)\neq 0$ and $cl_{T_2}cl_{T_1}(1-\lambda)\neq 0$, in (X,T_1,T_2). Hence, for the pairwise fuzzy dense set λ in (X,T_1,T_2), $cl_{T_1}cl_{T_2}(1-\lambda)\neq 0$ and $cl_{T_2}cl_{T_1}(1-\lambda)\neq 0$, implies that (X,T_1,T_2) is a pairwise fuzzy irresolvable space.

The following propositions give conditions, in terms of pairwise fuzzy dense set, for a fuzzy bitopological space to be a pairwise fuzzy irresolvable space.

Proposition 4.9 If $Bd_{12}(\lambda)\neq 1$ and $Bd_{21}(\lambda)\neq 1$, for each pairwise fuzzy dense set λ in a fuzzy bitopological space (X,T_1,T_2), then (X,T_1,T_2) is a pairwise fuzzy irresolvable space.

Proof: Let λ be a pairwise fuzzy dense set in (X, T_1, T_2). Then $cl_{T_1}cl_{T_2}(\lambda) = 1 = cl_{T_2}cl_{T_1}(\lambda)$, in (X, T_1, T_2). Now $Bd_{12}(\lambda) = cl_{T_1}cl_{T_2}(\lambda) \wedge cl_{T_1}cl_{T_2}(1-\lambda)$ and $Bd_{21}(\lambda) = cl_{T_2}cl_{T_1}(\lambda) \wedge cl_{T_2}cl_{T_1}(1-\lambda)$, in (X, T_1, T_2). Then $Bd_{12}(\lambda) = 1 \wedge cl_{T_1}cl_{T_2}(1-\lambda) = cl_{T_1}cl_{T_2}(1-\lambda)$ and $Bd_{21}(\lambda) = 1 \wedge cl_{T_2}cl_{T_1}(1-\lambda) = cl_{T_2}cl_{T_1}(1-\lambda)$, in (X, T_1, T_2). Since $Bd_{12}(\lambda) \neq 1$ and $Bd_{21}(\lambda) \neq 1$, $cl_{T_1}cl_{T_2}(1-\lambda) \neq 1$ and $cl_{T_2}cl_{T_1}(1-\lambda) \neq 1$, in (X, T_1, T_2). Hence, for the pairwise fuzzy dense set λ in (X, T_1, T_2), $cl_{T_1}cl_{T_2}(1-\lambda) \neq 1$ and $cl_{T_2}cl_{T_1}(1-\lambda) \neq 1$ in (X, T_1, T_2), implies that (X, T_1, T_2) is a pairwise fuzzy irresolvable space.

Proposition 4.10 If $int_{T_1}(Bd_{12}(\lambda)) = 0$ and $int_{T_2}(Bd_{21}(\lambda)) = 0$, for each pairwise fuzzy dense set λ in a fuzzy bitopological space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy irresolvable space.

Proof: Suppose that $int_{T_1}(Bd_{12}(\lambda)) = 0$ and $int_{T_2}(Bd_{21}(\lambda)) = 0$, for each pairwise fuzzy dense set λ in (X, T_1, T_2). Since λ is a pairwise fuzzy dense set in (X, T_1, T_2), $cl_{T_1}cl_{T_2}(\lambda) = 1 = cl_{T_2}cl_{T_1}(\lambda)$ in (X, T_1, T_2). Now $int_{T_1}(Bd_{12}(\lambda)) = 0$ implies that $int_{T_1}[cl_{T_1}cl_{T_2}(\lambda) \wedge cl_{T_1}cl_{T_2}(1-\lambda)] = 0$ in (X, T_1, T_2). Then $int_{T_1}[1 \wedge cl_{T_1}cl_{T_2}(1-\lambda)] = 0$ and hence $int_{T_1}[cl_{T_1}cl_{T_2}(1-\lambda)] = 0$ in (X, T_1, T_2). Then $cl_{T_1}cl_{T_2}(1-\lambda) \neq 1$ in (X, T_1, T_2). [Otherwise, if $cl_{T_1}cl_{T_2}(1-\lambda) = 1$ in (X, T_1, T_2), then $int_{T_1}[cl_{T_1}cl_{T_2}(1-\lambda)] = int_{T_1}(1) = 1 \neq 0$, a contradiction].

Also, $int_{T_2}(Bd_{21}(\lambda)) = 0$ in (X, T_1, T_2), implies that $int_{T_2}[cl_{T_2}cl_{T_1}(\lambda) \wedge cl_{T_2}cl_{T_1}(1-\lambda)] = 0$ in (X, T_1, T_2). Then $int_{T_2}[1 \wedge cl_{T_2}cl_{T_1}(1-\lambda)] = 0$ and hence $int_{T_2}[cl_{T_2}cl_{T_1}(1-\lambda)] = 0$ in (X, T_1, T_2). Then $cl_{T_2}cl_{T_1}(1-\lambda) \neq 1$ in (X, T_1, T_2). [Otherwise, if $cl_{T_2}cl_{T_1}(1-\lambda) = 1$ in (X, T_1, T_2), then $int_{T_2}[cl_{T_2}cl_{T_1}(1-\lambda)] = int_{T_2}(1) = 1 \neq 0$, a contradiction].

Thus, $cl_{T_1}cl_{T_2}(1-\lambda) \neq 1$ and $cl_{T_2}cl_{T_1}(1-\lambda) \neq 1$, for a pairwise fuzzy dense set λ in (X, T_1, T_2). Therefore, (X, T_1, T_2) is a pairwise fuzzy irresolvable space.

The following proposition gives a condition for a fuzzy bitopological space to be a pairwise fuzzy irresolvable space.

Proposition 4.11 If each pairwise fuzzy dense set is a pairwise fuzzy semi-open set in a fuzzy bitopological space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy irresolvable space.

Proof: Suppose that each pairwise fuzzy dense set is a pairwise fuzzy semi-open set in (X, T_1, T_2). Let λ $(\neq 0)$ be a pairwise fuzzy dense set in (X, T_1, T_2).

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Then by hypothesis, λ is pairwise fuzzy semi-open set in (X, T_1, T_2) and hence $\lambda \leq cl_{T_i}int_{T_j}(\lambda)$ $(i \neq j$ and $i, j = 1, 2)$. Then $int_{T_j}(\lambda) \neq 0$ in (X, T_1, T_2). [Otherwise, if $int_{T_j}(\lambda) = 0$ in (X, T_1, T_2), then $cl_{T_i}int_{T_j}(\lambda) = cl_{T_i}(0) = 0$, a contradiction to $\lambda \leq cl_{T_i}int_{T_j}(\lambda)$]. Thus, $int_{T_1}(\lambda) \neq 0$ and $int_{T_2}(\lambda) \neq 0$ in (X, T_1, T_2). Then, $int_{T_1}int_{T_2}(\lambda) \neq 0$ and $int_{T_2}int_{T_1}(\lambda) \neq 0$ in (X, T_1, T_2). [For, otherwise if $int_{T_1}int_{T_2}(\lambda) = 0$ and $int_{T_2}int_{T_1}(\lambda) = 0$ in (X, T_1, T_2), then $1 - int_{T_1}int_{T_2}(\lambda) = 1$ and $1 - int_{T_2}int_{T_1}(\lambda) = 1$ and hence it will be $cl_{T_1}cl_{T_2}(1 - \lambda) = 1$ and $cl_{T_2}cl_{T_1}(1 - \lambda) = 1$ in (X, T_1, T_2) and thus $1 - \lambda$ is a pairwise fuzzy dense set in (X, T_1, T_2). Then, by hypothesis, $1 - \lambda$ will be a pairwise fuzzy semi-open set and hence λ will be a pairwise fuzzy semi-closed set, a contradiction to λ being a pairwise fuzzy semi-open set in (X, T_1, T_2). Therefore, $int_{T_1}int_{T_2}(\lambda) \neq 0$ and $int_{T_2}int_{T_1}(\lambda) \neq 0$ in (X, T_1, T_2)]. Hence λ is a pairwise fuzzy faintly open set in (X, T_1, T_2). Thus, the pairwise fuzzy dense set λ is a pairwise fuzzy faintly open set in (X, T_1, T_2) and hence by proposition 7.3, (X, T_1, T_2) is a pairwise fuzzy irresolvable space.

Proposition 4.12 A fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy irresolvable space if and only if $\{1 - cl_{T_i}cl_{T_j}(\lambda) \mid int_{T_i}int_{T_j}(\lambda) = 0$ $(i \neq j$ and $i, j = 1, 2)$, for a fuzzy set λ defined on $X\}$ is a fuzzy filter on X.

Proof: Let $K = \{1 - cl_{T_i}cl_{T_j}(\lambda) \mid int_{T_i}int_{T_j}(\lambda) = 0$ $(i \neq j$ and $i, j = 1, 2)$, for a fuzzy set λ defined on $X\}$ be a fuzzy filter on X. It has to be proved that the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy irresolvable space. Assume the contrary. Suppose that (X, T_1, T_2) is a pairwise fuzzy resolvable space. Then, by theorem 2.15, there exists a fuzzy set μ defined on X such that $cl_{T_1}cl_{T_2}(\mu) = 1 = cl_{T_2}cl_{T_1}(\mu)$ and $int_{T_1}int_{T_2}(\mu) = 0 = int_{T_2}int_{T_1}(\mu)$ in (X, T_1, T_2).

Now $cl_{T_1}cl_{T_2}(\mu) = 1 = cl_{T_2}cl_{T_1}(\mu)$ in (X, T_1, T_2), implies that $1 - cl_{T_1}cl_{T_2}(\mu) = 0 = 1 - cl_{T_2}cl_{T_1}(\mu)$ in (X, T_1, T_2). Thus, $1 - cl_{T_i}cl_{T_j}(\mu) = 0$ $(i \neq j$ and $i, j = 1, 2)$ in (X, T_1, T_2).

Now $int_{T_i}int_{T_j}(\mu) = 0$ $(i \neq j$ and $i, j = 1, 2)$, implies that $1 - cl_{T_i}cl_{T_j}(\lambda) \in K$ and hence $0 \in K$. But this is a contradiction to K being a fuzzy filter on X in which $0 \notin K$. Therefore, (X, T_1, T_2) must be a pairwise fuzzy irresolvable space.

Conversely, suppose that (X, T_1, T_2) is a pairwise fuzzy irresolvable space. Consider collection $K = \{1 - cl_{T_i}cl_{T_j}(\lambda) \mid int_{T_i}int_{T_j}(\lambda) = 0$ $(i \neq j$ and $i, j = 1, 2)$ of fuzzy sets defined on X.

(i) First, it has to be proved that $0 \notin K$. Assume the contrary. Suppose that $0 \in K$. Then, $0 = 1 - cl_{T_i}cl_{T_j}(\delta)$ $(i \neq j$ and $i, j = 1, 2)$, for some fuzzy set δ

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defined on X such that $int_{T_i}int_{T_j}(\delta) = 0$ in (X, T_1, T_2). Now $1 - cl_{T_i}cl_{T_j}(\delta) = 0$, implies that $cl_{T_i}cl_{T_j}(\delta) = 1$ in (X, T_1, T_2) and hence δ is a pairwise fuzzy dense set in (X, T_1, T_2). Now $int_{T_i}int_{T_j}(\delta) = 0$, implies that $cl_{T_i}cl_{T_j}(1 - \delta) = 1 - int_{T_i}int_{T_j}(\delta) = 1 - 0 = 1$ and hence $1 - \delta$ is a pairwise fuzzy dense set in (X, T_1, T_2). Thus, there exists a pairwise fuzzy dense set δ (X, T_1, T_2) such that $1 - \delta$ is also a pairwise fuzzy dense set in (X, T_1, T_2). This will imply that (X, T_1, T_2) will be a pairwise fuzzy resolvable space, a contradiction. Hence the assumption that $0 \in K$ does not hold and therefore $0 \notin K$.

(ii) Let $\mu \in K$. Then, there exists a fuzzy set δ defined on X such that $\mu = 1 - cl_{T_i}cl_{T_j}(\delta)$ and $int_{T_i}int_{T_j}(\delta) = 0$ in (X, T_1, T_2). If η is a fuzzy set defined on X such that $\eta < \delta$, then $int_{T_i}int_{T_j}(\eta) < int_{T_i}int_{T_j}(\delta)$ and $cl_{T_i}cl_{T_j}(\eta) < cl_{T_i}cl_{T_j}(\delta)$ in (X, T_1, T_2). Then, $int_{T_i}int_{T_j}(\eta) < 0$ and $1 - cl_{T_i}cl_{T_j}(\eta) > 1 - cl_{T_i}cl_{T_j}(\delta)$ in (X, T_1, T_2). That is, $int_{T_i}int_{T_j}(\eta) = 0$ in (X, T_1, T_2). Let $\lambda = 1 - cl_{T_i}cl_{T_j}(\eta)$. Then, $\lambda > \mu$. Hence $int_{T_i}int_{T_j}(\eta) = 0$ in (X, T_1, T_2), implies that $1 - cl_{T_i}cl_{T_j}(\eta) \in K$. That is, $\lambda \in K$. Thus, if $\lambda > \mu$ and $\mu \in K$, then $\lambda \in K$.

(iii) Let $\lambda_1, \lambda_2 \in K$. Then, $\lambda_1 = 1 - cl_{T_i}cl_{T_j}(\delta_1)$; $int_{T_i}int_{T_j}(\delta_1) = 0$ and
$\lambda_2 = 1 - cl_{T_i}cl_{T_j}(\delta_2)$; $int_{T_i}int_{T_j}(\delta_2) = 0$ $(i \neq j$ and $i, j = 1, 2)$.

$$\begin{aligned} \text{Now } \lambda_1 \wedge \lambda_2 &= [1 - cl_{T_i}cl_{T_j}(\delta_1)] \wedge [1 - cl_{T_i}cl_{T_j}(\delta_2)] \\ &= 1 - [cl_{T_i}cl_{T_j}(\delta_1) \vee cl_{T_i}cl_{T_j}(\delta_2)] \end{aligned}$$

$$\begin{aligned} &= 1 - cl_{T_i}cl_{T_j}(\delta_1 \vee \delta_2) \\ \text{and hence } \lambda_1 \wedge \lambda_2 &= 1 - cl_{T_i}cl_{T_j}(\delta_1 \vee \delta_2) \longrightarrow (A) \quad (i \neq j \text{ and } i, j = 1, 2) \end{aligned}$$

in (X, T_1, T_2). If $int_{T_i}int_{T_j}(\delta_1 \vee \delta_2) = 0$, then $1 - int_{T_i}int_{T_j}(\delta_1 \vee \delta_2) = 1$ and hence $cl_{T_i}cl_{T_j}(1 - (\delta_1 \vee \delta_2)) = 1$ in (X, T_1, T_2). Then, $1 - (\delta_1 \vee \delta_2)$ is a pairwise fuzzy dense set in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy irresolvable space, $cl_{T_i}cl_{T_j}(1 - (1 - (\delta_1 \vee \delta_2))) \neq 1$, for the pairwise fuzzy dense set $1 - (\delta_1 \vee \delta_2)$ in (X, T_1, T_2). That is, $cl_{T_i}cl_{T_j}(\delta_1 \vee \delta_2) \neq 1$ and hence $1 - cl_{T_i}cl_{T_j}(\delta_1 \vee \delta_2) \neq 0$ in (X, T_1, T_2). Hence, $int_{T_i}int_{T_j}(\delta_1 \vee \delta_2) = 0$ and $1 - cl_{T_i}cl_{T_j}(\delta_1 \vee \delta_2) \neq 0$, implies that $1 - cl_{T_i}cl_{T_j}(\delta_1 \vee \delta_2) \in K$. Thus, from (A), we will have $\lambda_1 \wedge \lambda_2 \in K$, for the fuzzy sets $\lambda_1, \lambda_2 \in K$. Therefore, $\{1 - cl_{T_i}cl_{T_j}(\lambda) \mid int_{T_i}int_{T_j}(\lambda) = 0\}$ is a fuzzy filter on X.

5.Pairwise fuzzy open hereditarily irresolvable spaces

Definition 5.1 A fuzzy set λ defined on a set X is called a pairwise fuzzy somewhere dense set in a fuzzy bitopological space (X, T_1, T_2) if $int_{T_i}cl_{T_j}(\lambda) \neq 0$ $(i \neq j$ and $i, j =$

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$1,2)$ in (X,T_1,T_2). That is, λ is a pairwise fuzzy somewhere dense set in (X,T_1,T_2) if $int_{T_1}cl_{T_2}(\lambda) \neq 0$ and $int_{T_2}cl_{T_1}(\lambda) \neq 0$, in (X,T_1,T_2).

Example 5.2 Let $X = \{a,b,c\}$. The fuzzy sets λ, μ and δ are defined on X as follows:

$\lambda : X \to [0,1]$ is defined as $\lambda(a) = 0.25;\ \ \lambda(b) = 0;\ \ \lambda(c) = 0$

$\mu : X \to [0,1]$ is defined as $\mu(a) = 0.75;\ \ \mu(b) = 0.5;\ \ \mu(c) = 0$

$\delta : X \to [0,1]$ is defined as $\delta(a) = 0.8;\ \ \delta(b) = 0;\ \ \delta(c) = 1.$

Clearly, $T_1 = \{0,\lambda,\mu,1\}$ and $T_2 = \{0,\lambda,\delta,1\}$ are fuzzy topologies on X. On computation, $int_{T_1}cl_{T_2}(\lambda) = \mu$ and $int_{T_2}cl_{T_1}(\lambda) = \lambda$ in (X,T_1,T_2) and hence $int_{T_i}cl_{T_j}(\lambda) \neq 0$ $(i \neq j$ and $i,j = 1,2)$ in (X,T_1,T_2). Therefore, λ is a pairwise fuzzy somewhere dense set in (X,T_1,T_2). Also, $int_{T_1}cl_{T_2}(1-\delta) = int_{T_1}(1-\delta) = 0$ and $int_{T_2}cl_{T_1}(1-\delta) = int_{T_2}(1-\lambda) = \lambda$ in (X,T_1,T_2) and hence $1-\delta$ is not a pairwise fuzzy somewhere dense set in (X,T_1,T_2).

Proposition 5.3 If λ is a pairwise fuzzy somewhere dense set in a fuzzy bitopological space (X,T_1,T_2), then $cl_{T_i}int_{T_j}(1-\lambda) \neq 1$ $(i \neq j$ and $i,j = 1,2)$, in (X,T_1,T_2).

Proof: Let λ be a pairwise fuzzy somewhere dense set in a fuzzy bitopological space (X,T_1,T_2). Then $int_{T_i}cl_{T_j}(\lambda) \neq 0$ $(i \neq j$ and $i,j = 1,2)$ in (X,T_1,T_2). Now $1 - int_{T_i}cl_{T_j}(\lambda) \neq 1 - 0 = 1$ in (X,T_1,T_2). That is, $1 - int_{T_i}cl_{T_j}(\lambda) \neq 1$ in (X,T_1,T_2) and hence $cl_{T_i}int_{T_j}(1-\lambda) \neq 1$ in (X,T_1,T_2).

The following proposition ensures the existence of pairwise fuzzy somewhere dense sets in a fuzzy bitopological space.

Proposition 5.4 If λ is a non-zero pairwise fuzzy open set in a fuzzy bitopological space (X,T_1,T_2), then λ is a pairwise fuzzy somewhere dense set in (X,T_1,T_2).

Proof: Let $\lambda(\neq 0)$ be a pairwise fuzzy open set in (X,T_1,T_2). Then $int_{T_i}(\lambda) = \lambda$, $(i = 1,2)$ in (X,T_1,T_2). Now $\lambda \leq cl_{T_j}(\lambda)$, $(j = 1,2)$ in (X,T_1,T_2), implies that $int_{T_i}(\lambda) \leq int_{T_i}cl_{T_j}(\lambda)$ and hence $\lambda \leq int_{T_i}cl_{T_j}(\lambda)$ $(i \neq j$ and $i,j = 1,2)$ in (X,T_1,T_2). Then $int_{T_i}cl_{T_j}(\lambda) \neq 0$ in (X,T_1,T_2). Therefore, λ is a pairwise fuzzy somewhere dense set in (X,T_1,T_2).

*g.thangaraj@rediffmail.com, **vivekjsc@gmail.com

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Proposition 5.5 If μ is a pairwise fuzzy closed set in a fuzzy bitopological space (X, T_1, T_2), then $cl_{T_i}int_{T_j}(\mu) \neq 1$ $(i \neq j$ and $i, j = 1, 2)$ in (X, T_1, T_2).

Proof: Let μ be a pairwise fuzzy closed set in (X, T_1, T_2). Then $1-\mu$ is a pairwise fuzzy open set in (X, T_1, T_2) and hence, by proposition 5.4, $1-\mu$ is a pairwise fuzzy somewhere dense set in (X, T_1, T_2). Then, $int_{T_i}cl_{T_j}(1-\mu) \neq 0$ $(i \neq j$ and $i, j = 1, 2)$ in (X, T_1, T_2). This implies that $1 - cl_{T_i}int_{T_j}(\mu) \neq 0$ in (X, T_1, T_2). Therefore, $cl_{T_i}int_{T_j}(\mu) \neq 1$ $(i \neq j$ and $i, j = 1, 2)$ in (X, T_1, T_2).

Definition 5.6 A fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy open hereditarily irresolvable space if for each pairwise fuzzy somewhere dense set λ in (X, T_1, T_2), $int_{T_i}(\lambda) \neq 0$ $(i = 1, 2)$ in (X, T_1, T_2).

Example 5.7 Let $X = \{a, b, c\}$. The fuzzy sets λ, μ, δ, η and α are defined on X as follows:

$\lambda : X \to [0, 1]$ is defined as $\lambda(a) = 0.25$; $\lambda(b) = 0$; $\lambda(c) = 0$

$\mu : X \to [0, 1]$ is defined as $\mu(a) = 0.75$; $\mu(b) = 0.5$; $\mu(c) = 0$

$\delta : X \to [0, 1]$ is defined as $\delta(a) = 0.6$; $\delta(b) = 0$; $\delta(c) = 1$

$\eta : X \to [0, 1]$ is defined as $\eta(a) = 0.7$; $\eta(b) = 0.4$; $\eta(c) = 0.5$

$\alpha : X \to [0, 1]$ is defined as $\alpha(a) = 0.6$; $\alpha(b) = 0.4$; $\alpha(c) = 1$.

Clearly, $T_1 = \{0, \lambda, \mu, 1\}$ and $T_2 = \{0, \lambda, \delta, 1\}$ are fuzzy topologies on X. On computation, one can see that $int_{T_i}cl_{T_j}(\lambda) \neq 0$, $int_{T_i}cl_{T_j}(\mu) \neq 0$, $int_{T_i}cl_{T_j}(\delta) \neq 0$, $int_{T_i}cl_{T_j}(\eta) \neq 0$, $int_{T_i}cl_{T_j}(\alpha) \neq 0$, $int_{T_i}cl_{T_j}(1-\lambda) \neq 0$, $int_{T_i}cl_{T_j}(1-\mu) \neq 0$, $int_{T_i}cl_{T_j}(1-\delta) \neq 0$, $int_{T_i}cl_{T_j}(1-\eta) \neq 0$ and $int_{T_i}cl_{T_j}(1-\alpha) \neq 0$ $(i \neq j$ and $i, j = 1, 2)$ in (X, T_1, T_2) and hence λ, μ, δ, η, α, $1-\lambda$, $1-\mu$, $1-\delta$, $1-\eta$ and $1-\alpha$ are the pairwise fuzzy somewhere dense sets in (X, T_1, T_2). Now $int_{T_i}(\lambda) \neq 0$, $int_{T_i}(\mu) \neq 0$, $int_{T_i}(\delta) \neq 0$, $int_{T_i}(\eta) \neq 0$, $int_{T_i}(\alpha) \neq 0$, $int_{T_i}(1-\lambda) \neq 0$, $int_{T_i}(1-\mu) \neq 0$, $int_{T_i}(1-\delta) \neq 0$, $int_{T_i}(1-\eta) \neq 0$ and $int_{T_i}(1-\alpha) \neq 0$, $(i = 1, 2)$ in (X, T_1, T_2), implies that (X, T_1, T_2) is a pairwise fuzzy open hereditarily irresolvable space.

Example 5.8 Let $X = \{a, b, c\}$. The fuzzy sets λ, μ, δ and η are defined on X as follows:

$\lambda : X \to [0, 1]$ is defined as $\lambda(a) = 0.25$; $\lambda(b) = 0$; $\lambda(c) = 0$

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$\mu : X \to [0,1]$ is defined as $\mu(a) = 0.75;\quad \mu(b) = 0.5;\quad \mu(c) = 0$

$\delta : X \to [0,1]$ is defined as $\delta(a) = 0.8;\quad \delta(b) = 0;\quad \delta(c) = 1$

$\eta : X \to [0,1]$ is defined as $\eta(a) = 0.9;\quad \eta(b) = 0.4;\quad \eta(c) = 0.5.$

Clearly, $T_1 = \{0, \lambda, \mu, 1\}$ and $T_2 = \{0, \lambda, \delta, 1\}$ are fuzzy topologies on X. On computation, the pairwise fuzzy somewhere dense sets in (X, T_1, T_2) are λ, μ, $1 - \lambda$, $1 - \mu$ and $1 - \eta$. For the pairwise fuzzy somewhere dense set $1 - \eta$ in (X, T_1, T_2), $int_{T_i}(1 - \eta) = 0$ in (X, T_1, T_2) implies that (X, T_1, T_2) is not a pairwise fuzzy open hereditarily irresolvable space.

6.Characterizations of pairwise fuzzy open hereditarily irresolvable spaces

Proposition 6.1 If $cl_{T_1}(\lambda) = 1$ and $cl_{T_2}(\lambda) = 1$ in a pairwise fuzzy open hereditarily irresolvable space (X, T_1, T_2), then $int_{T_1}(\lambda) \neq 0$ and $int_{T_2}(\lambda) \neq 0$ in (X, T_1, T_2).

Proof: Suppose that $cl_{T_1}(\lambda) = 1$ and $cl_{T_2}(\lambda) = 1$ in (X, T_1, T_2). Then, $int_{T_2} cl_{T_1}(\lambda) = int_{T_2}(1) = 1 \neq 0$ and $int_{T_1} cl_{T_2}(\lambda) = int_{T_1}(1) = 1 \neq 0$ in (X, T_1, T_2). Hence $int_{T_i} cl_{T_j}(\lambda) \neq 0 \quad (i \neq j$ and $i, j = 1, 2)$ in (X, T_1, T_2). Then, λ is a pairwise fuzzy somewhere dense set in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy open hereditarily irresolvable space, for the pairwise fuzzy somewhere dense set λ in (X, T_1, T_2), $int_{T_i}(\lambda) \neq 0$ in (X, T_1, T_2). Thus, if $cl_{T_1}(\lambda) = 1$ and $cl_{T_2}(\lambda) = 1$ in a pairwise fuzzy open hereditarily irresolvable space (X, T_1, T_2), then $int_{T_1}(\lambda) \neq 0$ and $int_{T_2}(\lambda) \neq 0$ in (X, T_1, T_2).

Proposition 6.2 If $int_{T_1}(\lambda) = 0$ and $int_{T_2}(\lambda) = 0$ in a pairwise fuzzy open hereditarily irresolvable space (X, T_1, T_2), then $cl_{T_1}(\lambda) \neq 1$ and $cl_{T_2}(\lambda) \neq 1$ in (X, T_1, T_2).

Proof: Suppose that $int_{T_1}(\lambda) = 0$ and $int_{T_2}(\lambda) = 0$ in (X, T_1, T_2). Then $1 - int_{T_1}(\lambda) = 1$ and $1 - int_{T_2}(\lambda) = 1$ in (X, T_1, T_2) and hence $cl_{T_1}(1 - \lambda) = 1$ and $cl_{T_2}(1 - \lambda) = 1$ in (X, T_1, T_2). Then, by proposition 6.1, $int_{T_1}(1 - \lambda) \neq 0$ and $int_{T_2}(1 - \lambda) \neq 0$ in (X, T_1, T_2) and hence $1 - cl_{T_1}(\lambda) \neq 0$ and $1 - cl_{T_2}(\lambda) \neq 0$ in (X, T_1, T_2). Therefore, $cl_{T_1}(\lambda) \neq 1$ and $cl_{T_2}(\lambda) \neq 1$ in (X, T_1, T_2).

Proposition 6.3 If λ is a pairwise fuzzy dense set in a fuzzy bitopological space (X, T_1, T_2), then $1 - \lambda$ is a pairwise fuzzy somewhere dense set in (X, T_1, T_2).

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Proof: Let λ be a pairwise fuzzy dense set in (X, T_1, T_2). Then, $cl_{T_i}cl_{T_j}(\lambda) = 1$ $(i \neq j$ and $i, j = 1, 2)$ in (X, T_1, T_2). Now $int_{T_i}cl_{T_j}(1 - \lambda) = 1 - cl_{T_i}int_{T_j}(\lambda) > 1 - cl_{T_i}cl_{T_j}(\lambda) = 1 - 1 = 0$, in (X, T_1, T_2). That is, $int_{T_i}cl_{T_j}(1 - \lambda) > 0$ and hence $int_{T_i}cl_{T_j}(1 - \lambda) \neq 0$ in (X, T_1, T_2). Thus, $1 - \lambda$ is a pairwise fuzzy somewhere dense set in (X, T_1, T_2).

Proposition 6.4 If λ is a pairwise fuzzy dense set in a pairwise fuzzy open hereditarily irresolvable space (X, T_1, T_2), then $cl_{T_i}(\lambda) \neq 1$ $(i = 1, 2)$ in (X, T_1, T_2).

Proof: Let λ be a pairwise fuzzy dense set in (X, T_1, T_2). Then, $cl_{T_i}cl_{T_j}(\lambda) = 1$ $(i \neq j$ and $i, j = 1, 2)$ in (X, T_1, T_2). By proposition 6.3, $1 - \lambda$ is a pairwise fuzzy somewhere dense set in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy open hereditarily irresolvable space, for the pairwise fuzzy somewhere dense set $1 - \lambda$ in (X, T_1, T_2), $int_{T_i}(1 - \lambda) \neq 0$ in (X, T_1, T_2). Then, $1 - cl_{T_i}(\lambda) \neq 0$ in (X, T_1, T_2) and hence $cl_{T_i}(\lambda) \neq 1$ $(i = 1, 2)$ in (X, T_1, T_2). Thus, $cl_{T_i}cl_{T_j}(\lambda) = 1$ in a pairwise fuzzy open hereditarily irresolvable space (X, T_1, T_2) does not imply that $cl_{T_i}(\lambda) = 1$, in (X, T_1, T_2).

Proposition 6.5 If λ is a pairwise fuzzy dense set in a pairwise fuzzy open hereditarily irresolvable space (X, T_1, T_2), then there exist a T_1-fuzzy closed set δ and a T_2-fuzzy closed set η such that $cl_{T_2}(\delta) = 1$ and $cl_{T_1}(\eta) = 1$ in (X, T_1, T_2).

Proof: Let λ be a pairwise fuzzy dense set in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy open hereditarily irresolvable space, by proposition 6.4, $cl_{T_i}(\lambda) \neq 1$ $(i = 1, 2)$, in (X, T_1, T_2). Let $cl_{T_1}(\lambda) = \delta$ and $cl_{T_2}(\lambda) = \eta$ in (X, T_1, T_2). Since $cl_{T_1}cl_{T_2}(\lambda) = 1$ and $cl_{T_2}cl_{T_1}(\lambda) = 1$ in (X, T_1, T_2), $cl_{T_1}(\eta) = 1$ and $cl_{T_2}(\delta) = 1$ in (X, T_1, T_2). Thus, δ is a T_1-fuzzy closed and T_2-fuzzy dense set and η is a T_2-fuzzy closed and T_1-fuzzy dense set in (X, T_1, T_2).

Proposition 6.6 If λ is a pairwise fuzzy dense set in a pairwise fuzzy open hereditarily irresolvable space (X, T_1, T_2), then $int_{T_i}(1 - \lambda) \neq 0$ $(i = 1, 2)$ in (X, T_1, T_2).

Proof: Let λ be a pairwise fuzzy dense set in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy open hereditarily irresolvable space, by proposition 6.4, $cl_{T_i}(\lambda) \neq 1$ $(i = 1, 2)$, in (X, T_1, T_2). Then, $1 - cl_{T_i}(\lambda) \neq 0$, in (X, T_1, T_2) and $int_{T_i}(1 - \lambda) \neq 0$, in (X, T_1, T_2).

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Proposition 6.7 If $int_{T_i}int_{T_j}(\lambda) = 0$ $(i \neq j$ and $i, j = 1, 2)$ in a pairwise fuzzy open hereditarily irresolvable space (X, T_1, T_2), then $int_{T_i}(\lambda) \neq 0$ in (X, T_1, T_2).

Proof: Suppose that $int_{T_i}int_{T_j}(\lambda) = 0$ $(i \neq j$ and $i, j = 1, 2)$ in (X, T_1, T_2). Then, $cl_{T_i}cl_{T_j}(1 - \lambda) = 1 - int_{T_i}int_{T_j}(\lambda) = 1 - 0 = 1$, in (X, T_1, T_2) and hence $1 - \lambda$ is a pairwise fuzzy dense set in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy open hereditarily irresolvable space, by proposition 6.4, $cl_{T_i}(1-\lambda) \neq 1$ in (X, T_1, T_2). Then, $1 - int_{T_i}(\lambda) \neq 1$ in (X, T_1, T_2), implies that $int_{T_i}(\lambda) \neq 0$ in (X, T_1, T_2).

Proposition 6.8 If $cl_{T_i}int_{T_j}(\mu) \neq 1$ $(i \neq j$ and $i, j = 1, 2)$ in a pairwise fuzzy open hereditarily irresolvable space (X, T_1, T_2), then $cl_{T_i}(\mu) \neq 1$ in (X, T_1, T_2).

Proof: Suppose that $cl_{T_i}int_{T_j}(\mu) \neq 1$ $(i \neq j$ and $i, j = 1, 2)$ in a pairwise fuzzy open hereditarily irresolvable space (X, T_1, T_2). Then, $1 - cl_{T_i}int_{T_j}(\mu) \neq 0$, in (X, T_1, T_2) and hence $int_{T_i}cl_{T_j}(1 - \mu) \neq 0$, in (X, T_1, T_2). Then $1 - \mu$ is a pairwise fuzzy somewhere dense set in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy open hereditarily irresolvable space, for the pairwise fuzzy somewhere dense set $1 - \mu$ in (X, T_1, T_2), $int_{T_i}(1 - \mu) \neq 0$ in (X, T_1, T_2). Then $1 - cl_{T_i}(\mu) \neq 0$, in (X, T_1, T_2) and hence $cl_{T_i}(\mu) \neq 1$ in (X, T_1, T_2).

Proposition 6.9 If $\lambda(\neq 0)$ is a pairwise fuzzy regular open set in a pairwise fuzzy open hereditarily irresolvable space (X, T_1, T_2), then $int_{T_i}(\lambda) \neq 0$ $(i = 1, 2)$ in (X, T_1, T_2).

Proof: Let λ be a pairwise fuzzy regular open set in (X, T_1, T_2). Then, $int_{T_i}cl_{T_j}(\lambda) = \lambda$, $(i \neq j$ and $i, j = 1, 2)$ in (X, T_1, T_2). Now $\lambda \neq 0$ in (X, T_1, T_2), implies that $int_{T_i}cl_{T_j}(\lambda) \neq 0$, in (X, T_1, T_2) and hence λ is a pairwise fuzzy somewhere dense set in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy open hereditarily irresolvable space, for the pairwise fuzzy somewhere dense set λ in (X, T_1, T_2), $int_{T_i}(\lambda) \neq 0$ $(i = 1, 2)$ in (X, T_1, T_2).

Proposition 6.10 If $\lambda(\neq 0)$ is a pairwise fuzzy pre-open set in a pairwise fuzzy open hereditarily irresolvable space (X, T_1, T_2), then $int_{T_i}(\lambda) \neq 0$ $(i = 1, 2)$ in (X, T_1, T_2).

Proof: Suppose that λ $(\neq 0)$ is a pairwise fuzzy pre-open set in (X, T_1, T_2). Then, $\lambda \leq int_{T_i}cl_{T_j}(\lambda)$ $(i \neq j$ and $i, j = 1, 2)$ in (X, T_1, T_2) and hence $int_{T_i}cl_{T_j}(\lambda) \neq 0$, in (X, T_1, T_2). Thus, λ is a pairwise fuzzy somewhere dense set in (X, T_1, T_2). Since

(X, T_1, T_2) is a pairwise fuzzy open hereditarily irresolvable space, for the pairwise fuzzy somewhere dense set λ in (X, T_1, T_2), $int_{T_i}(\lambda) \neq 0 \ \ (i = 1, 2)$ in (X, T_1, T_2).

7.Pairwise fuzzy open hereditarily irresolvable spaces and other fuzzy bitopological spaces

Proposition 7.1 If a fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy open hereditarily irresolvable space, then (X, T_1, T_2) is a pairwise fuzzy irresolvable space.

Proof: Let λ be a pairwise fuzzy dense set in (X, T_1, T_2). Then, $cl_{T_i}cl_{T_j}(\lambda) = 1$, $(i \neq j$ and $i, j = 1, 2)$ in (X, T_1, T_2) and hence $1 - cl_{T_i}cl_{T_j}(\lambda) = 1 - 1 = 0$, in (X, T_1, T_2). This implies that $int_{T_i}int_{T_j}(1-\lambda) = 0$, in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy open hereditarily irresolvable space, by proposition 6.7, $int_{T_i}(\lambda) \neq 0$, in (X, T_1, T_2). Now, $int_{T_i}int_{T_j}(\lambda) \leq int_{T_j}(\lambda)$, in (X, T_1, T_2), implies that $1 - int_{T_i}int_{T_j}(\lambda) \geq 1 - int_{T_j}(\lambda)$, in (X, T_1, T_2) and $cl_{T_i}cl_{T_j}(1-\lambda) \neq 1$, in (X, T_1, T_2). Thus, for a pairwise fuzzy dense set λ in (X, T_1, T_2), $cl_{T_i}cl_{T_j}(1-\lambda) \neq 1 \ \ (i \neq j$ and $i, j = 1, 2)$ in (X, T_1, T_2), implies that (X, T_1, T_2) is a pairwise fuzzy irresolvable space.

Remark: The converse of the above proposition need not be true. That is, a pairwise fuzzy irresolvable space need not be a pairwise fuzzy open hereditarily irresolvable space. For, consider the following example:

Example 7.2 Let $X = \{a, b, c\}$. The fuzzy sets λ, μ, δ and η are defined on X as follows:

$\lambda : X \to [0, 1]$ is defined as $\lambda(a) = 0.25; \ \ \lambda(b) = 0; \ \ \lambda(c) = 0$

$\mu : X \to [0, 1]$ is defined as $\mu(a) = 0.75; \ \ \mu(b) = 0.5; \ \ \mu(c) = 0$

$\delta : X \to [0, 1]$ is defined as $\delta(a) = 0.8; \ \ \delta(b) = 0; \ \ \delta(c) = 1$

$\eta : X \to [0, 1]$ is defined as $\eta(a) = 0.9; \ \ \eta(b) = 0.4; \ \ \eta(c) = 0.5$.

Clearly, $T_1 = \{0, \lambda, \mu, 1\}$ and $T_2 = \{0, \lambda, \delta, 1\}$ are fuzzy topologies on X. On computation, one can see that $cl_{T_1}cl_{T_2}(\delta) = 1 = cl_{T_2}cl_{T_1}(\delta)$ in (X, T_1, T_2) and hence δ is a pairwise fuzzy dense set in (X, T_1, T_2). But, $cl_{T_1}cl_{T_2}(1-\delta) = cl_{T_1}(1-\delta) = 1-\lambda \neq 1$ and $cl_{T_2}cl_{T_1}(1 - \delta) = cl_{T_2}(1 - \lambda) = 1 - \lambda \neq 1$ and hence, for the pairwise fuzzy dense set δ in (X, T_1, T_2), $cl_{T_1}cl_{T_2}(1 - \delta) \neq 1$ and $cl_{T_2}cl_{T_1}(1 - \delta) \neq 1$ in (X, T_1, T_2). This shows that (X, T_1, T_2) is a pairwise fuzzy irresolvable space. But, $int_{T_1}cl_{T_2}(1 - \eta) = int_{T_1}(1 - \lambda) = \mu \neq 0$ and $int_{T_2}cl_{T_1}(1 - \eta) = int_{T_2}(1 - \lambda) = \lambda \neq 0$ in (X, T_1, T_2) and hence η is a pairwise fuzzy somewhere dense set in (X, T_1, T_2). But $int_{T_1}(1 - \eta) = 0$

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and $int_{T_2}(1-\eta)=0$, implies that (X,T_1,T_2) is not a pairwise fuzzy open hereditarily irresolvable space.

Proposition 7.3 A fuzzy bitopological space (X,T_1,T_2) is a pairwise fuzzy irresolvable space if and only if each pairwise fuzzy dense set is a pairwise fuzzy faintly open set in (X,T_1,T_2).

Proof: Let (X,T_1,T_2) be a pairwise fuzzy irresolvable space and λ be a pairwise fuzzy dense set in (X,T_1,T_2). Since (X,T_1,T_2) is a pairwise fuzzy irresolvable space, for the pairwise fuzzy dense set λ in (X,T_1,T_2), $cl_{T_1}cl_{T_2}(1-\lambda)\neq 1$ and $cl_{T_2}cl_{T_1}(1-\lambda)\neq 1$ in (X,T_1,T_2). Then $1-int_{T_1}int_{T_2}(\lambda)\neq 1$ and $1-int_{T_2}int_{T_1}(\lambda)\neq 1$ in (X,T_1,T_2) and hence $int_{T_1}int_{T_2}(\lambda)\neq 0$ and $int_{T_2}int_{T_1}(\lambda)\neq 0$ in (X,T_1,T_2). Therefore λ is a pairwise fuzzy faintly open set in (X,T_1,T_2).

Conversely, let each pairwise fuzzy dense set λ be a pairwise fuzzy faintly open set in (X,T_1,T_2). Then $\lambda\neq 0$. [otherwise $\lambda=0$ will imply that $cl_{T_1}cl_{T_2}(\lambda)=cl_{T_1}cl_{T_2}(0)=0\neq 1$ and $cl_{T_2}cl_{T_1}(\lambda)=cl_{T_2}cl_{T_1}(0)=0\neq 1$, a contradiction to λ being a pairwise fuzzy dense set in (X,T_1,T_2).] It has to be proved that (X,T_1,T_2) is a pairwise fuzzy irresolvable space. Assume the contrary. Suppose that (X,T_1,T_2) is a pairwise fuzzy resolvable space. Then, there exists a pairwise fuzzy dense set μ in (X,T_1,T_2) such that $cl_{T_1}cl_{T_2}(1-\mu)=1$ and $cl_{T_2}cl_{T_1}(1-\mu)=1$ in (X,T_1,T_2) and hence $1-int_{T_1}int_{T_2}(\mu)=1$ and $1-int_{T_2}int_{T_1}(\mu)=1$ in (X,T_1,T_2). Then $int_{T_1}int_{T_2}(\mu)=0$ and $int_{T_2}int_{T_1}(\mu)=0$ in (X,T_1,T_2), a contraction to the hypothesis that each pairwise fuzzy dense set is a pairwise fuzzy faintly open set (X,T_1,T_2). Hence, the assumption that (X,T_1,T_2) is a pairwise fuzzy resolvable space, does not hold and therefore (X,T_1,T_2) is a pairwise fuzzy irresolvable space,

Proposition 7.4 If λ is a pairwise fuzzy dense set in a pairwise fuzzy open hereditarily irresolvable space (X,T_1,T_2), then $int_{T_i}int_{T_j}(\lambda)\neq 0$, $(i\neq j$ and $i,j=1,2)$ in (X,T_1,T_2).

Proof: Let λ be a pairwise fuzzy dense set in (X,T_1,T_2). Since (X,T_1,T_2) is a pairwise fuzzy open hereditarily irresolvable space, by proposition 7.1, (X,T_1,T_2) is a pairwise fuzzy irresolvable space. Then, by proposition 7.3, the pairwise fuzzy dense set λ is a pairwise fuzzy faintly open set in (X,T_1,T_2) and hence $int_{T_i}int_{T_j}(\lambda)\neq 0$, $(i\neq j$ and $i,j=1,2)$ in (X,T_1,T_2). Thus, for a fuzzy set λ with $cl_{T_i}cl_{T_j}(\lambda)=1$, in a pairwise fuzzy open hereditarily irresolvable space (X,T_1,T_2), $int_{T_i}int_{T_j}(\lambda)\neq 0$, $(i\neq j$ and $i,j=1,2)$ in (X,T_1,T_2).

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Proposition 7.5 If a fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy irresolvable space, then $\lambda_1 \wedge \lambda_2 \neq 0$, for any two pairwise fuzzy dense sets λ_1 and λ_2 in (X, T_1, T_2).

Proof: Let (X, T_1, T_2) be a pairwise fuzzy irresolvable space and λ_1 and λ_2 be any two pairwise fuzzy dense sets in (X, T_1, T_2). Then $cl_{T_1}cl_{T_2}(\lambda_1) = 1 = cl_{T_2}cl_{T_1}(\lambda_1)$ and $cl_{T_1}cl_{T_2}(\lambda_2) = 1 = cl_{T_2}cl_{T_1}(\lambda_2)$ It has to be proved that $\lambda_1 \wedge \lambda_2 \neq 0$ in (X, T_1, T_2). Assume the contrary. Suppose that $\lambda_1 \wedge \lambda_2 = 0$ in (X, T_1, T_2). Then λ_1 and λ_2 are any two disjoint fuzzy sets in (X, T_1, T_2) and hence by definition 2.12, $\lambda_1 \leq (1 - \lambda_2)$ in (X, T_1, T_2). Then $cl_{T_1}cl_{T_2}(\lambda_1) \leq cl_{T_1}cl_{T_2}(1 - \lambda_2)$ and $cl_{T_2}cl_{T_1}(\lambda_1) \leq cl_{T_2}cl_{T_1}(1 - \lambda_2)$ in (X, T_1, T_2) and hence $1 \leq cl_{T_1}cl_{T_2}(1 - \lambda_2)$ and $1 \leq cl_{T_2}cl_{T_1}(1 - \lambda_2)$ in (X, T_1, T_2). That is, $cl_{T_1}cl_{T_2}(1 - \lambda_2) = 1$ and $cl_{T_2}cl_{T_1}(1 - \lambda_2) = 1$ in (X, T_1, T_2). Thus, there exists a pairwise fuzzy dense set λ_2 in (X, T_1, T_2) such that $cl_{T_1}cl_{T_2}(1 - \lambda_2) = 1 = cl_{T_2}cl_{T_1}(1 - \lambda_2)$. This means that (X, T_1, T_2) is a pairwise fuzzy resolvable space, a contradiction to (X, T_1, T_2) being a pairwise fuzzy irresolvable space. Hence, the assumption that $\lambda_1 \wedge \lambda_2 = 0$ in (X, T_1, T_2) does not hold. Therefore, $\lambda_1 \wedge \lambda_2 \neq 0$, for any two pairwise fuzzy dense sets λ_1 and λ_2 in a pairwise fuzzy irresolvable space (X, T_1, T_2).

Proposition 7.6 If λ_1 and λ_2 are any two pairwise fuzzy dense sets in a pairwise fuzzy open hereditarily irresolvable space (X, T_1, T_2), then $\lambda_1 \wedge \lambda_2 \neq 0$ in (X, T_1, T_2).

Proof: Let λ_1 and λ_2 be any two pairwise fuzzy dense sets in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy open hereditarily irresolvable space, by proposition 7.1, (X, T_1, T_2) is a pairwise fuzzy irresolvable space. Then, by proposition 7.5, $\lambda_1 \wedge \lambda_2 \neq 0$, for the pairwise fuzzy dense sets λ_1 and λ_2 in (X, T_1, T_2).

Proposition 7.7 If a fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy submaximal space, then (X, T_1, T_2) is a pairwise fuzzy open hereditarily irresolvable space.

Proof: Let λ be a pairwise fuzzy somewhere dense set in (X, T_1, T_2). Then, $int_{T_i}cl_{T_j}(\lambda) \neq 0$, $(i \neq j$ and $i, j = 1, 2)$ in (X, T_1, T_2). It has to be proved that $int_{T_i}(\lambda) \neq 0$ $(i = 1, 2)$ in (X, T_1, T_2). Assume the contrary. Suppose that $int_{T_i}(\lambda) = 0$ $(i = 1, 2)$ in (X, T_1, T_2). Then, $int_{T_j}int_{T_i}(\lambda) = int_{T_j}(0) = 0$ in (X, T_1, T_2) and hence $int_{T_j}int_{T_i}(\lambda) = 0$ in (X, T_1, T_2). This implies that $1 - int_{T_j}int_{T_i}(\lambda) = 1 - 0 = 1$ in (X, T_1, T_2). Then, $cl_{T_j}cl_{T_i}(1 - \lambda) = 1$ in (X, T_1, T_2) and hence $1 - \lambda$ is a pairwise fuzzy

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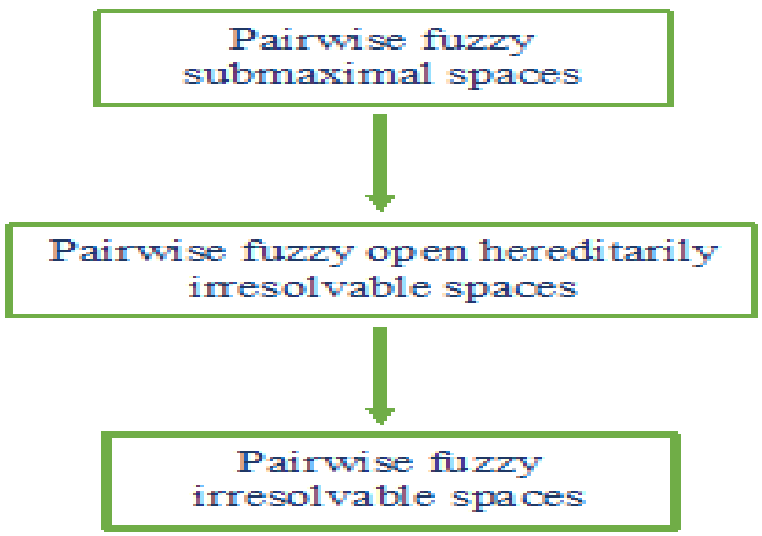
dense set in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy submaximal space, the pairwise fuzzy dense set $1 - \lambda$ will be a pairwise fuzzy open set in (X, T_1, T_2). Then, λ will be a pairwise fuzzy closed set in (X, T_1, T_2) and hence $cl_{T_i}(\lambda) = \lambda \ \ (i = 1, 2)$ in (X, T_1, T_2). This implies that $int_{T_i} cl_{T_j}(\lambda) = int_{T_i}(\lambda) = 0 \ \ (i \neq j$ and $i, j = 1, 2)$, a contradiction. Hence, it must be that $int_{T_i}(\lambda) \neq 0 \ \ (i = 1, 2)$, for the pairwise fuzzy somewhere dense set λ in (X, T_1, T_2). Therefore, (X, T_1, T_2) is a pairwise fuzzy open hereditarily irresolvable space.

Remark: Inter-relations between pairwise fuzzy open hereditarily irresolvable spaces, pairwise fuzzy submaximal spaces and pairwise fuzzy irresolvable spaces can be summarized as follows:



8.Conclusion

Several characterizations of pairwise fuzzy irresolvable spaces and pairwise fuzzy open hereditarily irresolvable spaces have studied and established in this paper.

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