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Discrete Fourier Transform Arising from Generalized Difference Operator

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Abstract

In this paper, we obtain discrete Fourier transform and its properties for certain functions using the inverse of generalized difference operator. Suitable examples verified by MATLAB are inserted to illustrate the main results.

Key words: Generalized Discrete Fourier Transform, Polynomial factorials and Generalized difference operator.

AMS classification: 65T50, 26D05, 39A70.

1. Introduction

Digital Signal Processing (DSP) has revolutionized many areas in science and engineering such as space, medical, commercial, military, industrial and communication. DSP is made effectively possible by Fourier Transform (FT) and Discrete Fourier Transform (DFT) which decomposes signals into sinusoids. In [6, 7], the forward complex DFT, written in polar form, is given by:

$$X(k) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \tag{1}$$

and it takes different labels depending on the nature of $x[n]$. The DFT $X(k) = \frac{1}{N}\Delta^{-1}x(n)e^{-j2\pi kn/N}\Big|_0^N$ follows from the basic difference identity $\Delta^{-1}x(k)\Big|_0^N = \sum_{n=0}^{N-1} x(n)$ in [1]. Let ℓ be the time between two successive signals. Replacing Δ^{-1} by Δ_ℓ^{-1}, integer n by real t and the sequence $x(n)$ by a function $x(t)$, we get a

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Generalized Discrete Fourier Transform (GDFT)

$$X(k) = \frac{\ell}{N}\Delta_\ell^{-1}x(t)e^{-j2\pi kt/N}\Big|_0^N. \tag{2}$$

When $\ell = 1$ and $\ell \to 0$ the GDFT becomes DFT given in (1) and FT defined as $g(k) = \int\limits_{-\infty}^{\infty} e^{-j2\pi kt/N} f(t)dt$ respectively [2, 3]. In this paper, we have successfully arrived at a new transform GDFT for signals of algebraic and geometric functions employing the basic theory of Δ_ℓ and its inverse. We have also additionally provided certain properties of GDFT.

2. Preliminaries

In [4, 5] the authors introduce $k_\ell^{(m)} = k(k-\ell)(k-2\ell)\cdots(k-(m-1)\ell)$, the operator Δ_ℓ as $\Delta_\ell u(k) = u(k+\ell) - u(k)$ and its inverse by

$$if\ \Delta_\ell v(k) = u(k),\ then\ v(k)\ =\ \Delta_\ell^{-1}u(k). \tag{3}$$

Using Stirling numbers of first kind s_r^m and second kind S_r^m, we obtain

$$(i)k_\ell^{(m)} = \sum_{r=1}^{m} s_r^m \ell^{m-r}k^r, (ii)k^m = \sum_{r=1}^{m} S_r^m \ell^{m-r}k_\ell^{(r)}, (iii)\Delta_\ell k_\ell^{(m)} = (m\ell)k_\ell^{(m-1)} \tag{4}$$

$$(iv)\Delta_\ell^{-1}k_\ell^{(m)} = \frac{k_\ell^{(m+1)}}{\ell(m+1)}(v)\Delta_\ell^{-1}k^m = \sum_{r=1}^{m}\frac{S_r^m\ell^{m-r}k_\ell^{(r)}}{(r+1)\ell}(vi)\Delta_\ell^{-1}e^{isk} = \frac{e^{isk}}{(e^{is\ell}-1)}. \tag{5}$$

Lemma 2.1 [4] Let $\ell > 0$, $k \in (\ell,\ \infty)$. Then the inverse of product of two functions is

$$\Delta_\ell^{-1}(u(k)w(k)) = u(k)\Delta_\ell^{-1}w(k) - \Delta_\ell^{-1}(\Delta_\ell^{-1}w(k+\ell)\Delta_\ell u(k)). \tag{6}$$

Lemma 2.2 [5] If $u(k)$ is a bounded function on $[a,b]$ and $\ell = \dfrac{b-a}{M}$, then

$$\Delta_\ell^{-1}u(k)\Big|_a^b = \sum_{r=1}^{M} u(b-r\ell) = \sum_{r=0}^{M-1} u(a+r\ell). \tag{7}$$

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3. Properties of Generalized Discrete Fourier Transform

In this section we derive closed form solution of the generalized difference equation $\Delta_\ell v(k) = u(k)$ and we obtain some properties of generalized discrete Fourier complex, sine and cosine transforms.

Theorem 3.1 Let $k \in (-\infty, \infty)$ and $\ell > 0$. Then we have

$$\Delta_\ell^{-1}(k_\ell^{(m)} e^{\pm isk}) = \sum_{j=0}^{m} \frac{(-1)^j (m)_1^{(j)} \ell^j k_\ell^{(m-j)} e^{\pm is(k+j\ell)}}{(e^{\pm is\ell} - 1)^{j+1}} \tag{8}$$

Proof: Taking $u(k) = k_\ell^{(1)}$, $w(k) = e^{isk}$ in (6) and using (4), we get

$$\Delta_\ell^{-1}(k_\ell^{(1)} e^{isk}) = \frac{k_\ell^{(1)} e^{isk}}{(e^{is\ell} - 1)} - \frac{\ell e^{is(k+\ell)}}{(e^{is\ell} - 1)^2}.$$

Taking $u(k) = k_\ell^{(2)}$, $w(k) = e^{isk}$ in (6), and using (4), we get

$$\Delta_\ell^{-1}(k_\ell^{(2)} e^{isk}) = \frac{k_\ell^{(2)} e^{isk}}{(e^{is\ell} - 1)} - \frac{2\ell k_\ell^{(1)} e^{is(k+\ell)}}{(e^{is\ell} - 1)^2} + \frac{2\ell^2 e^{is(k+2\ell)}}{(e^{is\ell} - 1)^3}.$$

which can be expressed as

$$\Delta_\ell^{-1}(k_\ell^{(2)} e^{isk}) = \sum_{j=0}^{2} \frac{(-1)^j (2)_1^{(j)} \ell^j k_\ell^{(2-j)} e^{is(k+j\ell)}}{(e^{is\ell} - 1)^{j+1}}.$$

Now (8) follows by continuing the above process and then replacing i by $-i$.

Theorem 3.2 Let $k \in (-\infty, \infty)$and $\ell > 0$. If $e^{is\ell} \neq 1$, then we have

$$\Delta_\ell^{-1}(k^n e^{\pm isk}) = \sum_{m=0}^{n} S_m^n \ell^{n-m} \sum_{j=0}^{m} \frac{(-1)^j (m)_1^{(j)} \ell^j k_\ell^{(m-j)} e^{\pm is(k+j\ell)}}{(e^{\pm is\ell} - 1)^{j+1}} \tag{9}$$

Proof: The proof follows from second term of (4) and (8).

Theorem 3.3 Let $k \in (-\infty, \infty)$ and $\ell > 0$. If $a^\ell e^{\pm in\ell} \neq 1$, then

$$\Delta_\ell^{-1}(a^k e^{\pm ink}) = \frac{a^k e^{\pm ink}}{(a^\ell e^{\pm in\ell} - 1)} \tag{10}$$

Proof: Since $\Delta_\ell a^k e^{\pm ink} = a^{k+\ell} e^{\pm in(k+\ell)} - a^k e^{\pm ink}$, the proof follows from (3).

Definition 3.4 The generalized discrete Fourier transform (GDFT) of $u(k)$ is defined

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as

$$F(u(k)) = U(s) = \ell\Delta_\ell^{-1} u(k) e^{isk}\Big|_{k=-\infty}^{\infty} \tag{11}$$

and the inverse generalized discrete Fourier transform of $U(s)$ is given by

$$u(k) = \frac{\ell}{2\pi}\Delta_\ell^{-1} U(s) e^{-isk}\Big|_{s=-\infty}^{\infty} \tag{12}$$

Similarly the generalized discrete Fourier sine and cosine transforms of $u(k)$ are defined as

$$F_s(u(k)) = U_s(s) = \ell\Delta_\ell^{-1} u(k)\sin sk\Big|_{k=0}^{\infty} \tag{13}$$

$$F_c(u(k)) = U_c(s) = \ell\Delta_\ell^{-1} u(k)\cos sk\Big|_{k=0}^{\infty}. \tag{14}$$

The inverse generalized discrete Fourier sine and cosine transforms of the above are respectively given by

$$u(k) = \frac{2\ell}{\pi}\Delta_\ell^{-1} U_s(s)\sin sk\Big|_{s=0}^{\infty} \tag{15}$$

$$u(k) = \frac{2\ell}{\pi}\Delta_\ell^{-1} U_c(s)\cos sk\Big|_{s=0}^{\infty}. \tag{16}$$

Let a and b are constants. From the linearity Δ_ℓ^{-1} and the Definition 3.4, we can easily obtain the following linear, change of scale and shifting properties.

Property 3.5 If $F(u(k)) = U(s)$ and $F(v(k)) = V(s)$, then

$$(i)\ F(au(k) + bv(k)) = aU(s) + bV(s), \tag{17}$$

$$(ii)\ If\ \ F(u(k)) = U(s)\ \ then\ \ F(u(ak)) = \frac{1}{a}U\Big(\frac{s}{a}\Big), \tag{18}$$

$$(iii)\ If\ \ F_s(u(k)) = U_s(s)\ \ then\ \ F_s(u(ak)) = \frac{1}{a}U_s\Big(\frac{s}{a}\Big), \tag{19}$$

$$(iv)\ If\ \ F_c(u(k)) = U_c(s)\ \ then\ \ F_c(u(ak)) = \frac{1}{a}U_c\Big(\frac{s}{a}\Big) \tag{20}$$

Example 3.6 Take $u(k) = k^2$ for $0 < k < 4$. From (20), (13) and (9), we arrive

$$U_c(s) = \frac{\ell}{2}\left\{\frac{16e^{i4s}}{(e^{is\ell}-1)} - \frac{\ell^2(e^{is(4+\ell)-e^{is\ell}})}{(e^{is\ell}-1)^2} - \frac{2\ell}{(e^{is\ell}-1)}\Big(\frac{4e^{is(4+\ell)}}{(e^{is\ell}-1)} - \frac{\ell(e^{is(4+2\ell)-e^{is2\ell}})}{(e^{is\ell}-1)^2}\Big)\right\}$$
$$+\frac{\ell}{2}\left\{\frac{16e^{-i4s}}{(e^{-is\ell}-1)} - \frac{\ell^2(e^{-is(4+\ell)-e^{-is\ell}})}{(e^{-is\ell}-1)^2} - \frac{2\ell}{(e^{-is\ell}-1)}\Big(\frac{4e^{-is(4+\ell)}}{(e^{-is\ell}-1)} - \frac{\ell(e^{-is(4+2\ell)-e^{-is2\ell}})}{(e^{-is\ell}-1)^2}\Big)\right\}$$

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In particular, when $\ell \to 0$ we get $U_c(s) = \dfrac{16\sin 4s}{s} + \dfrac{5\cos 4s - 1}{s^2} - \dfrac{2\sin 4s}{s^3}$ which yields $F_c((ak)^2) = \dfrac{1}{a}\Big(\dfrac{16\sin(4s/a)}{(s/a)} + \dfrac{5\cos(4s/a) - 1}{(s/a)^2} - \dfrac{2\sin(4s/a)}{(s/a)^3}\Big)$.

Property 3.7 If $F(u(k)) = U(s)$, then $F(u(k-a)) = e^{isa}U(s)$

Proof: From (11), we have $F(u(k-a)) = \ell\Delta_\ell^{-1}u(k-a)e^{isk}\big|_{k=-\infty}^{\infty}$.
Now the proof follows by substituting $k-a$ by t.

Example 3.8 Take $u(k) = e^{ik}$ for $-1 < k < 1$. From the Property (3.7) and (11), we obtain the Fourier transform $F(e^{ik}) = \ell\Big(\dfrac{e^{i(s+1)} - e^{-i(s+1)}}{(e^{i(s+1)\ell} - 1)}\Big)$.

In particular, when $\ell \to 0$, we get $U(s) = \dfrac{2\sin(s+1)}{(s+1)}$ and $F(u(k-a)) = e^{isa}\dfrac{2\sin(s+1)}{(s+1)}$.

Theorem 3.9 Modulation theorem:

$$(i)\ If\ \ F(u(k)) = U(s)\ \ then\ \ F(u(k)\cos ak) = \frac{1}{2}[U(s+a) + U(s-a)] \tag{21}$$

$$(ii)\ If\ \ F_s(u(k)) = U_s(s)\ \ then\ \ F_s(u(k)\cos ak) = \frac{1}{2}[U_s(s+a) + U_s(s-a)] \tag{22}$$

$$(iii)\ If\ \ F_s(u(k)) = U_s(s)\ \ then\ \ F_c(u(k)\sin ak) = \frac{1}{2}[U_s(s+a) - U_s(s-a)] \tag{23}$$

$$(iv)\ If\ \ F_c(u(k)) = U_c(s)\ \ then\ \ F_s(u(k)\sin ak) = \frac{1}{2}[U_c(s-a) - U_c(s+a)] \tag{24}$$

Proof: From (11), we get $F(u(k)\cos ak) = \ell\Delta_\ell^{-1}u(k)e^{isk}\Big(\dfrac{e^{iak} + e^{-iak}}{2}\Big)\Big|_{k=-\infty}^{\infty}$, and (13), we have $F_s(u(k)\cos ak) = \ell\Delta_\ell^{-1}u(k)\dfrac{1}{2}(\sin(s+a)k + \sin(s-a)k)\Big|_{k=0}^{\infty}$, which yield (21) and (22). Similarly, we can get the proof of (23) and (24).

Example 3.10 Taking $u(k) = a^k$ for $|k| \le 3$ and $u(k) = 0$ otherwise, then from (21) and (11), we have $F(a^k) = U(s) = \ell\Delta_\ell^{-1}a^k e^{isk}\big|_{k=-3}^{3} = \ell\Big(\dfrac{a^3e^{i3s} - a^{-3}e^{-i3s}}{a^\ell e^{is\ell} - 1}\Big)$.

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In particular, when $\ell \to 0$, we get $U(s) = \dfrac{1}{isa^3}(\cos 3s(a^6-1) + i\sin 3s(a^6+1))$. Which yield

$$F(a^k\cos ak) = \frac{1}{2}\left\{\frac{1}{i(s+a)a^3}(\cos 3(s+a)(a^6-1) + i\sin 3(s+a)(a^6+1)) + \frac{1}{i(s-a)a^3}(\cos 3(s-a)(a^6-1) + i\sin 3(s-a)(a^6+1))\right\}$$

Similarly taking $u(k) = k^3$ in (22) for $0 < k < 5$ and from (13), we have

$$F_s(k^3) = U_s(s) = \ell\Delta_\ell^{-1}k^3\sin sk\Big|_{k=0}^{5} = \ell\Delta_\ell^{-1}k^3\Big(\frac{e^{isk}-e^{-isk}}{2i}\Big)\Big|_{k=0}^{5}$$

Now applying (9), we derive

$$U_s(s) = \frac{\ell}{2i}\left\{\frac{5e^{i5s}}{(e^{is\ell}-1)} - \frac{\ell(75e^{is(5+\ell)} + 15\ell e^{is(5+\ell)} + \ell^2e^{is(5+\ell)} - \ell^2e^{is\ell})}{(e^{is\ell}-1)^2} - \frac{\ell^2(30e^{is(5+2\ell)} - 3\ell e^{is(5+2\ell)} + 3\ell e^{i2s\ell})}{(e^{is\ell}-1)^3} + \frac{6\ell^3(e^{is(5+3\ell)} - 3e^{i3s\ell})}{(e^{is\ell}-1)^4}\right\}$$
$$-\frac{\ell}{2i}\left\{\frac{5e^{-i5s}}{(e^{-is\ell}-1)} - \frac{\ell(75e^{-is(5+\ell)} + 15\ell e^{-is(5+\ell)} + \ell^2e^{-is(5+\ell)} - \ell^2e^{-is\ell})}{(e^{-is\ell}-1)^2} - \frac{\ell^2(30e^{-is(5+2\ell)} - 3\ell e^{-is(5+2\ell)} + 3\ell e^{-i2s\ell})}{(e^{-is\ell}-1)^3} + \frac{6\ell^3(e^{-is(5+3\ell)} - 3e^{-i3s\ell})}{(e^{-is\ell}-1)^4}\right\}.$$

When $\ell \to 0$, we get $U_s(s) = \dfrac{1}{s^4}(6\sin 5s - 30s\cos 5s + 75s^2\sin 5s - 5s^3\cos 5s)$

and
$$F_s(k^3\cos ak) = \frac{1}{2}\Big(\frac{1}{(s+a)^4}(6\sin 5(s+a) - 30(s+a)\cos 5(s+a) + 75(s+a)^2\sin 5(s+a) - 5(s+a)^3\cos 5(s+a)) + \frac{1}{(s-a)^4}(6\sin 5(s-a) - 30(s-a)\cos 5(s-a) + 75(s-a)^2\sin 5(s-a) - 5(s-a)^3\cos 5(s-a))\Big).$$

In the following example, we analyse the DFT ($\ell = 1$), GDFT ($\ell = 0.5$) and FT for the input signal $u(k) = k_\ell^{(2)}$ using MATLAB.

Example 3.11 Take $u(k) = k_\ell^{(2)}$ for $-2 < k < 2$. Then from (18), we get

$$F(k_\ell^{(2)}) = U(s) = \ell\Delta_\ell^{-1}k_\ell^{(2)}e^{isk}\Big|_{k=-2}^{2} = \ell\left\{\frac{(2)_\ell^{(2)}e^{i2s}}{(e^{is\ell}-1)} - \frac{2\ell(2)_\ell^{(1)}e^{is(2+\ell)}}{(e^{is\ell}-1)^2} + \frac{2\ell^2e^{is(2+2\ell)}}{(e^{is\ell}-1)^3} - \frac{(-2)_\ell^{(2)}e^{-i2s}}{(e^{is\ell}-1)} + \frac{2\ell(-2)_\ell^{(1)}e^{is(-2+\ell)}}{(e^{is\ell}-1)^2} - \frac{2\ell^2e^{is(-2+2\ell)}}{(e^{is\ell}-1)^3}\right\}.$$

Here we provide MATLAB coding for verification of DFT.
U(s)=symsum((2-r)*(1-r)*exp(i*2*(2-r)),r,1,4)
$=(2*exp(i*4)./(exp(i*2)-1)) - (4*exp(i*6)./(exp(i*2)-1).\hat{}2) + (2*exp(i*8)./(exp(i*2)-1).\hat{}3) - (6*exp(-1*i*4)./(exp(i*2)-1)) - (4*exp(-1*i*2)./(exp(i*2)-1).\hat{}2) - (2./(exp(i*2)-1).\hat{}3)$

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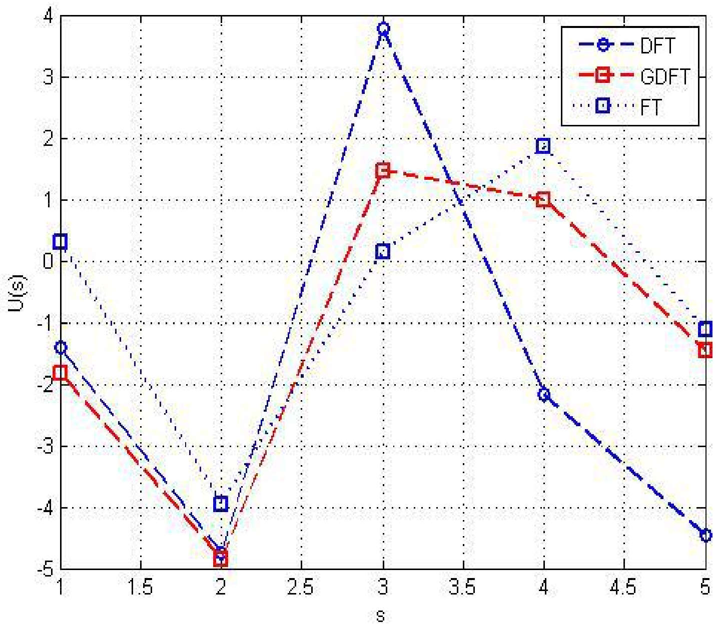
When $\ell \to 0$, we get FT as $U(s) = \dfrac{1}{s^3}(8s^2 \sin 2s + 8s \cos 2s - 4 \sin 2s)$
and $F((ak)_\ell^{(2)}) = \dfrac{a^2}{s^3}(8(s/a)^2 \sin 2(s/a) + 8(s/a) \cos 2(s/a) - 4 \sin 2(s/a))$.



4. Conclusion:

From the diagram we find that both DFT and FT can be obtained by GDFT. When the Fourier transform does not exist, we can apply generalized discrete Fourier transform from the above findings and get several applications in the field of digital signal process.

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