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Stability of a Functional equation originating from sum of first ℓ natural numbers in Intuitionistic Fuzzy Banach Space and Algebras using Direct and Fixed Point Methods

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Abstract

In this paper, the authors achieve the generalized Ulam - Hyers stability of a functional equation

$$\mathcal{W}\left(\sum_{j=1}^{\ell} j\ w_j\right) = \sum_{j=1}^{\ell} \left(j\ \mathcal{W}(w_j)\right), \qquad \ell \geq 1$$

which is originating from sum of first ℓ natural numbers via two substitutions in Intuitionistic Fuzzy Banach Space and Algebras using Direct and Fixed Point Methods.

Key words: Additive functional equations, Ulam - Hyers stability, Ulam - Hyers - Rassias stability, Ulam - Gavruta - Rassias stability, Ulam - JMRassias stability, Intuitionistic Fuzzy Banach space, Intuitionistic Fuzzy Banach algebra, fixed point.

AMS classification: 39B52, 32B72, 32B82

1.Introduction

For the period of latter eight decades, the stability problems of several functional equations have been extensively investigated by number of authors [1, 2, 27, 28, 40, 44, 47, 51]. The terminology generalized Ulam - Hyers stability originates from these chronological backgrounds. These terminologies are also applied to the case of other functional equations. For more detailed definitions of such terminologies, one

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can refer to [10, 22, 23, 29, 30, 31, 45]. One of the supreme renowned functional equations is the additive functional equation

$$f(x+y) = f(x) + f(y). \tag{1}$$

In 1821, it was first solved by A.L. Cauchy in the class of continuous real-valued functions. It is often called an additive Cauchy functional equation in honor of Cauchy (see [31]). The theory of additive functional equations is frequently applied to the development of theories of other functional equations. The solution and stability of the following various additive functional equations

$$f(2x-y) + f(x-2y) = 3f(x) - 3f(y), \tag{2}$$

$$f(x+y-2z) + f(2x+2y-z) = 3f(x) + 3f(y) - 3f(z), \tag{3}$$

$$f(m(x+y)-2mz) + f(2m(x+y)-mz) = 3m[f(x)+f(y)-f(z)] \ m \geq 1, \tag{4}$$

$$f\left(a\sum_{i=1}^{n-1} x_i - 2ax_n\right) + f\left(2a\sum_{i=1}^{n-1} x_i - ax_n\right) = 3a\left(\sum_{i=1}^{n-1} f(x_i) - f(x_n)\right) \ n \geq 3, \tag{5}$$

$$f(2x \pm y \pm z) = f(x \pm y) + f(x \pm z) \tag{6}$$

$$f(qx \pm y \pm z) = f(x \pm y) + f(x \pm z) + (q-2)f(x), \quad q \geq 2 \tag{7}$$

$$f(x) + f(x) = f(2x) \tag{8}$$

$$f(y) = \frac{f(y+z) + f(y-z)}{2} \tag{9}$$

$$f(x) = \sum_{\ell=1}^{n} \left(\frac{f(x+\ell y_\ell) + f(x-\ell y_\ell)}{2\ \ell}\right) \tag{10}$$

$$f\left(\frac{\sum\limits_{k=1}^{N} x_k}{N}\right) = \frac{1}{N}\sum_{k=1}^{N} f(x_k) \tag{11}$$

were discussed by D.O. Lee [25], K. Ravi, M. Arunkumar [46], M. Arunkumar [3, 4, 5, 6, 7, 10], J. M. Rassias et.al., [43]. The generalized Ulam - Hyers stability of several functional equations in various Banach spaces including Intuitionistic Fuzzy Banach spaces and Algebras were investigated in [8, 9, 11, 12, 15, 16, 17, 18, 19, 21, 13, 14, 43].

Let us recall the alternative of fixed point theorem to prove the stability results.

Theorem 1.1 [32] (The alternative of fixed point) Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $T : X \to X$ with

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Lipschitz constant L. Then, for each given element $x \in X$, either
(F_1) $\quad d(T^n x, T^{n+1}x) = \infty \quad \forall \ n \geq 0,$
or
(F_2) there exists a natural number n_0 such that:
$(FPC1)$ $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;
$(FPC2)$The sequence $(T^n x)$ is convergent to a fixed point y^* of T
$(FPC3)$ y^* is the unique fixed point of T in the set $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;
$(FPC4)$ $d(y^*, y) \leq \frac{1}{1-L} \ d(y, Ty)$ for all $y \in Y$.

In this paper, the authors achieve the generalized Ulam - Hyers stability of a functional equation

$$\mathcal{W}\left(\sum_{j=1}^{\ell} j \ w_j\right) = \sum_{j=1}^{\ell} \left(j \ \mathcal{W}(w_j)\right), \qquad \ell \geq 1 \tag{12}$$

which is originating from sum of first ℓ natural numbers via two substitutions in Intuitionistic Fuzzy Banach Space and Algebras using Direct and Fixed Point Methods.

2. Definitions and Notations of Intuitionistic Fuzzy Banach Space

Now, we recall the basic definitions and notations in the setting of intuitionistic fuzzy normed space given in [48].

Definition 2.1 A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be continuous t-norm if $*$ satisfies the following conditions:

(1) $*$ is commutative and associative;
(2) $*$ is continuous;
(3) $a * 1 = a$ for all $a \in [0,1]$;
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Definition 2.2 A binary operation $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be continuous t-conorm if $\diamond$ satisfies the following conditions:

(1') $\diamond$ is commutative and associative;
(2') $\diamond$ is continuous;
(3') $a \diamond 0 = a$ for all $a \in [0,1]$;
(4') $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

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Using the notions of continuous t-norm and t-conorm, Saadati and Park [48] introduced the concept of intuitionistic fuzzy normed space as follows:

Definition 2.3 The five-tuple $(X, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy normed space (for short, IFNS) if X is a vector space, $*$ is a continuous η-norm, $\diamond$ is a continuous $\eta-$ conorm, and μ, ν are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions. For every $x, y \in X$ and $s, t > 0$

(IFN1) $\mu(x,\eta) + \nu(x,\eta) \leq 1$,
(IFN2) $\mu(x,\eta) > 0$,
(IFN3) $\mu(x,\eta) = 1$, if and only if $x = 0$.
(IFN4) $\mu(dx,\eta) = \mu\left(x, \frac{\eta}{d}\right)$ for each $d \neq 0$,
(IFN5) $\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s)$,
(IFN6) $\mu(x,\cdot) : (0,\infty) \to [0,1]$ is continuous,
(IFN7) $\lim\limits_{t\to\infty} \mu(x,\eta) = 1$ and $\lim\limits_{t\to 0} \mu(x,\eta) = 0$,
(IFN8) $\nu(x,\eta) < 1$,
(IFN9) $\nu(x,\eta) = 0$, if and only if $x = 0$.
(IFN10) $\nu(dx,\eta) = \nu\left(x, \frac{\eta}{d}\right)$ for each $d \neq 0$,
(IFN11) $\nu(x,t) \diamond \nu(y,s) \geq \nu(x+y, t+s)$,
(IFN12) $\nu(x,\cdot) : (0,\infty) \to [0,1]$ is continuous,
(IFN13) $\lim\limits_{t\to\infty} \nu(x,\eta) = 0$ and $\lim\limits_{t\to 0} \nu(x,\eta) = 1$.

In this case, (μ, ν) is called an intuitionistic fuzzy norm.

Example 2.4 Let $(X, \|\cdot\|)$ be a normed space. Let $a*b = ab$ and $a \diamond d = \min\{a+b, 1\}$ for all $a, b \in [0,1]$. For all $x \in X$ and every $t > 0$, consider

$$\mu(x,t) = \begin{cases} \frac{t}{t+\|x\|} & if \quad t > 0; \\ 0 & if \quad t \leq 0; \end{cases} \qquad and \qquad \nu(x,t) = \begin{cases} \frac{\|x\|}{t+\|x\|} & if \quad t > 0; \\ 0 & if \quad t \leq 0. \end{cases}$$

Then $(X, \mu, \nu, *, \diamond)$ is an IFN-space.

Definition 2.5 Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then, a sequence $x = \{x_k\}$ is said to be intuitionistic fuzzy convergent to a point $L \in X$ if

$$\lim \quad \mu(x_k - L, t) = 1 \quad and \quad \lim \quad \nu(x_k - L, t) = 0$$

for all $\eta > 0$. In this case, we write

$$x_k \xrightarrow{IF} L \qquad as \qquad k \to \infty$$

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Definition 2.6 Let $(X, \mu, \nu, *, \diamond)$ be an IFN-space. Then, $x = \{x_k\}$ is said to be intuitionistic fuzzy Cauchy sequence if

$$\mu\left(x_{k+p} - x_k, t\ \right) = 1 \qquad and \qquad \nu\left(x_{k+p} - x_k, t\ \right) = 0$$

for all $\eta > 0$, and $p = 1, 2 \cdots$.

Definition 2.7 Let $(X, \mu, \nu, *, \diamond)$ be an IFN-space. Then $(X, \mu, \nu, *, \diamond)$ is said to be complete if every intuitionistic fuzzy Cauchy sequence in $(X, \mu, \nu, *, \diamond)$ is intuitionistic fuzzy convergent $(X, \mu, \nu, *, \diamond)$.

3. Stability In Intuitionistic Fuzzy Banach Space

In this section, the generalized Ulam - Hyers stability of the functional equation (12) in Intuitionistic Fuzzy Banach space is established. To prove stability results, let us take $\mathcal{N}_1$ be an Intuitionistic Fuzzy normed space and $\mathcal{N}_2$ be an Intuitionistic Fuzzy Banach space.

3.1 Hyers Method of (12)

Theorem 3.1 If $\mathcal{W} : \mathcal{N}_1 \to \mathcal{N}_2$ is a function satisfying the inequalities

$$\left.\begin{array}{l} \mu\left(\mathcal{W}\left(\sum_{j=1}^{\ell}\ j\ w_j\right) - \sum_{j=1}^{\ell}\left(j\ \mathcal{W}(w_j)\right), \eta\ \right) \geq \mu'\left(W\left(w_1, w_2, \cdots, w_\ell\right), \eta\ \right) \\ \nu\left(\mathcal{W}\left(\sum_{j=1}^{\ell}\ j\ w_j\right) - \sum_{j=1}^{\ell}\left(j\ \mathcal{W}(w_j)\right), \eta\ \right) \leq \nu'\left(W\left(w_1, w_2, \cdots, w_\ell\right), \eta\ \right) \end{array}\right\} \quad (13)$$

where $W : \mathcal{N}_1^{\ell} \longrightarrow (0, 1]$ with the conditions

$$\left.\begin{array}{l} \mu'\left(W\left(\gamma^{\alpha\beta} w_1, \gamma^{\alpha\beta} w_2, \cdots, \gamma^{\alpha\beta} w_\ell\right), \eta\ \right) \geq \mu'\left(\omega^{\alpha\beta} W\left(w_1, w_2, \cdots, w_\ell\right), \eta\right) \\ \nu'\left(W\left(\gamma^{\alpha\beta} w_1, \gamma^{\alpha\beta} w_2, \cdots, \gamma^{\alpha\beta} w_\ell\right), \eta\ \right) \leq \nu'\left(\omega^{\alpha\beta} W\left(w_1, w_2, \cdots, w_\ell\right), \eta\right) \end{array}\right\} \quad (14)$$

and

$$\left.\begin{array}{l} \lim_{\alpha\to\infty} \mu'\left(W\left(\gamma^{\alpha\beta} w_1, \gamma^{\alpha\beta} w_2, \cdots, \gamma^{\alpha\beta} w_\ell\right), \gamma^{\alpha\beta}\eta\right) = 1 \\ \lim_{\alpha\to\infty} \nu'\left(W\left(\gamma^{\alpha\beta} w_1, \gamma^{\alpha\beta} w_2, \cdots, \gamma^{\alpha\beta} w_\ell\right), \gamma^{\alpha\beta}\eta\right) = 0 \end{array}\right\} \quad (15)$$

for all $w_1, w_2, \cdots, w_\ell \in \mathcal{N}_1$ and all $\eta > 0$ with

$$\beta = \pm 1 \qquad \text{and} \qquad 0 < \left(\frac{w}{\gamma}\right)^{\beta} < 1. \quad (16)$$

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Then there exists a unique additive mapping $\Psi(w) : \mathcal{N}_1 \longrightarrow \mathcal{N}_2$ given by

$$\left.\begin{array}{l} \lim\limits_{\alpha\to\infty} \mu\left(\frac{\mathcal{W}(\gamma^\alpha w)}{\gamma^\alpha} - \Psi(w), \eta\right) = 1, \\ \lim\limits_{\alpha\to\infty} \nu\left(\frac{\mathcal{W}(\gamma^\alpha w)}{\gamma^\alpha} - \Psi(w), \eta\right) = 0 \end{array}\right\} \tag{17}$$

and satisfying the functional equation (12) and

$$\left.\begin{array}{l} \mu\left(\Psi(w) - \mathcal{W}(w), \eta\right) \geq \mu'\left(W\left(w, w, \cdots, w\right), |\gamma - \omega|\eta\right) \\ \nu\left(\Psi(w) - \mathcal{W}(w), \eta\right) \leq \nu'\left(W\left(w, w, \cdots, w\right), |\gamma - \omega|\eta\right) \end{array}\right\} \tag{18}$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$.

Proof. **Case (i):** Rechanging $w_1 = w_2 = \cdots = w_\ell = w$ in (13), one can arrive

$$\left.\begin{array}{l} \mu\left(\mathcal{W}\left(\left(\sum_{j=1}^{\ell}(j)\right)\ w\right) - \left(\sum_{j=1}^{\ell}(j)\right)\mathcal{W}(w), \eta\right) \geq \mu'\left(W\left(w, w, \cdots, w\right), \eta\right) \\ \nu\left(\mathcal{W}\left(\left(\sum_{j=1}^{\ell}(j)\right)\ w\right) - \left(\sum_{j=1}^{\ell}(j)\right)\mathcal{W}(w), \eta\right) \leq \nu'\left(W\left(w, w, \cdots, w\right), \eta\right) \end{array}\right\} \tag{19}$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$. Define $\sum_{j=1}^{\ell}(j) = 1 + 2 + 3 + \cdots + \ell = \frac{\ell(\ell+1)}{2} = \gamma$ in (19), one can have

$$\left.\begin{array}{l} \mu\left(\mathcal{W}\left(\gamma\ w\right) - \gamma\mathcal{W}(w), \eta\right) \geq \mu'\left(W\left(w, w, \cdots, w\right), \eta\right) \\ \nu\left(\mathcal{W}\left(\gamma\ w\right) - \gamma\mathcal{W}(w), \eta\right) \leq \nu'\left(W\left(w, w, \cdots, w\right), \eta\right) \end{array}\right\} \tag{20}$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$. It follows from (IFN4) and (IFN10) in (20), one can see

$$\left.\begin{array}{l} \mu\left(\dfrac{\mathcal{W}(\gamma w)}{\gamma} - \mathcal{W}(w), \dfrac{\eta}{\gamma}\right) \geq \mu'\left(W\left(w, w, \cdots, w\right), \eta\right) \\ \nu\left(\dfrac{\mathcal{W}(\gamma w)}{\gamma} - \mathcal{W}(w), \dfrac{\eta}{\gamma}\right) \leq \nu'\left(W\left(w, w, \cdots, w\right), \eta\right) \end{array}\right\} \tag{21}$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$. Putting w by $\gamma^\alpha w$ in (21), one can get

$$\left.\begin{array}{l} \mu\left(\dfrac{\mathcal{W}(\gamma^{\alpha+1} w)}{\gamma} - \mathcal{W}(\gamma^\alpha w), \dfrac{\eta}{\gamma}\right) \geq \mu'\left(W(\gamma^\alpha w, \gamma^\alpha w, \cdots, \gamma^\alpha w), \eta\right) \\ \nu\left(\dfrac{\mathcal{W}(\gamma^{\alpha+1} w)}{\gamma} - \mathcal{W}(\gamma^\alpha w), \dfrac{\eta}{\gamma}\right) \leq \nu'\left(W(\gamma^\alpha w, \gamma^\alpha w, \cdots, \gamma^\alpha w), \eta\right) \end{array}\right\} \tag{22}$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$. Using (IFN4), (IFN10) and (14) in (22), one can obtain

$$\left.\begin{array}{l} \mu\Big(\dfrac{\mathcal{W}(\gamma^{\alpha+1}w)}{\gamma^{(\alpha+1)}} - \dfrac{\mathcal{W}(\gamma^{\alpha}w)}{\gamma^{\alpha}}, \dfrac{\eta}{\gamma\cdot\gamma^{\alpha}}\Big) \geq \mu'\left(W(w,w,\cdots,w), \dfrac{\eta}{\omega^{\alpha}}\right) \\ \nu\Big(\dfrac{\mathcal{W}(\gamma^{\alpha+1}w)}{\gamma^{(\alpha+1)}} - \dfrac{\mathcal{W}(\gamma^{\alpha}w)}{\gamma^{\alpha}}, \dfrac{\eta}{\gamma\cdot\gamma^{\alpha}}\Big) \leq \nu'\left(W(w,w,\cdots,w), \dfrac{\eta}{\omega^{\alpha}}\right) \end{array}\right\} \quad (23)$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$. Interchanging η into $\omega^{\alpha}\eta$ in (23), one can find

$$\left.\begin{array}{l} \mu\left(\dfrac{\mathcal{W}(\gamma^{\alpha+1}w)}{\gamma^{(\alpha+1)}} - \dfrac{\mathcal{W}(\gamma^{\alpha}w)}{\gamma^{\alpha}}, \dfrac{\eta\cdot\omega^{\alpha}}{\gamma\cdot\gamma^{\alpha}}\right) \geq \mu'\left(W(w,w,\cdots,w), \eta\right) \\ \nu\left(\dfrac{\mathcal{W}(\gamma^{\alpha+1}w)}{\gamma^{(\alpha+1)}} - \dfrac{\mathcal{W}(\gamma^{\alpha}w)}{\gamma^{\alpha}}, \dfrac{\eta\cdot\omega^{\alpha}}{\gamma\cdot\gamma^{\alpha}}\right) \leq \nu'\left(W(w,w,\cdots,w), \eta\right) \end{array}\right\} \quad (24)$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$. One can easy to verify that

$$\frac{\mathcal{W}(\gamma^{\alpha}w)}{\gamma^{\alpha}} - \mathcal{W}(w) = \sum_{\rho=0}^{\alpha-1} \frac{\mathcal{W}(\gamma^{\rho+1}w)}{\gamma^{(\rho+1)}} - \frac{\mathcal{W}(\gamma^{\rho}w)}{\gamma^{\rho}} \quad (25)$$

for all $w \in \mathcal{N}_1$. From (25) one can achieve from (24) that

$$\left.\begin{array}{l} \mu\left(\dfrac{\mathcal{W}(\gamma^{\alpha}w)}{\gamma^{\alpha}} - \mathcal{W}(w), \dfrac{\eta}{\gamma}\cdot\displaystyle\sum_{\rho=0}^{\alpha-1}\dfrac{\omega^{\rho}}{\gamma^{\rho}}\right) = \mu\left(\displaystyle\sum_{\rho=0}^{\alpha-1}\dfrac{\mathcal{W}(\gamma^{\rho+1}w)}{\gamma^{(\rho+1)}} - \dfrac{\mathcal{W}(\gamma^{\rho}w)}{\gamma^{\rho}}, \dfrac{\eta}{\gamma}\cdot\displaystyle\sum_{\rho=0}^{\alpha-1}\dfrac{\omega^{\rho}}{\gamma^{\rho}}\right) \\ \nu\left(\dfrac{\mathcal{W}(\gamma^{\alpha}w)}{\gamma^{\alpha}} - \mathcal{W}(w), \dfrac{\eta}{\gamma}\cdot\displaystyle\sum_{\rho=0}^{\alpha-1}\dfrac{\omega^{\rho}}{\gamma^{\rho}}\right) = \nu\left(\displaystyle\sum_{\rho=0}^{\alpha-1}\dfrac{\mathcal{W}(\gamma^{\rho+1}w)}{\gamma^{(\rho+1)}} - \dfrac{\mathcal{W}(\gamma^{\rho}w)}{\gamma^{\rho}}, \dfrac{\eta}{\gamma}\cdot\displaystyle\sum_{\rho=0}^{\alpha-1}\dfrac{\omega^{\rho}}{\gamma^{\rho}}\right) \end{array}\right\} \quad (26)$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$. Using (IFNS5) and (IFNA11) in (26), one can observe

$$\left.\begin{array}{l} \mu\left(\dfrac{\mathcal{W}(\gamma^{\alpha}w)}{\gamma^{\alpha}} - \mathcal{W}(w), \dfrac{\eta}{\gamma}\cdot\displaystyle\sum_{\rho=0}^{\alpha-1}\dfrac{\omega^{\rho}}{\gamma^{\rho}}\right) \geq \displaystyle\prod_{\rho=0}^{\alpha-1}\mu\left(\dfrac{\mathcal{W}(\gamma^{\rho+1}w)}{\gamma^{(\rho+1)}} - \dfrac{\mathcal{W}(\gamma^{\rho}w)}{\gamma^{\rho}}, \dfrac{\eta}{\gamma}\cdot\dfrac{\omega^{\rho}}{\gamma^{\rho}}\right) \\ \nu\left(\dfrac{\mathcal{W}(\gamma^{\alpha}w)}{\gamma^{\alpha}} - \mathcal{W}(w), \dfrac{\eta}{\gamma}\cdot\displaystyle\sum_{\rho=0}^{\alpha-1}\dfrac{\omega^{\rho}}{\gamma^{\rho}}\right) \leq \displaystyle\prod_{\rho=0}^{\alpha-1}\nu\left(\dfrac{\mathcal{W}(\gamma^{\rho+1}w)}{\gamma^{(\rho+1)}} - \dfrac{\mathcal{W}(\gamma^{\rho}w)}{\gamma^{\rho}}, \dfrac{\eta}{\gamma}\cdot\dfrac{\omega^{\rho}}{\gamma^{\rho}}\right) \end{array}\right\} \quad (27)$$

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where $\prod_{\rho=0}^{\alpha-1} c_j = c_1 * c_2 * \cdots * c_\alpha$ *and* $\coprod_{\rho=0}^{\alpha-1} d_j = d_1 \diamond d_2 \diamond \cdots \diamond d_\alpha$ for all $w \in \mathcal{N}_1$ and all $\eta > 0$. It follows from (27) and (24) that

$$\left.\begin{aligned} \mu\left(\frac{\mathcal{W}(\gamma^\alpha w)}{\gamma^\alpha} - \mathcal{W}(w), \frac{\eta}{\gamma}\cdot\sum_{\rho=0}^{\alpha-1}\frac{\omega^\rho}{\gamma^\rho}\right) \geq \prod_{\rho=0}^{\alpha-1} \mu'\left(W(w,w,\cdots,w),\eta\ \right) = \mu'\left(W(w,w,\cdots,w),\eta\right) \\ \nu\left(\frac{\mathcal{W}(\gamma^\alpha w)}{\gamma^\alpha} - \mathcal{W}(w), \frac{\eta}{\gamma}\cdot\sum_{\rho=0}^{\alpha-1}\frac{\omega^\rho}{\gamma^\rho}\right) \leq \coprod_{\rho=0}^{\alpha-1} \nu'\left(W(w,w,\cdots,w),\eta\ \right) = \nu'\left(W(w,w,\cdots,w),\eta\right) \end{aligned}\right\} \tag{28}$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$. Setting w by $\gamma^\beta w$ in (28) and using (15), (IFN4), (IFN10), one can achieve

$$\left.\begin{aligned} \mu\left(\frac{\mathcal{W}(\gamma^\alpha\cdot\gamma^\beta w)}{\gamma^\alpha\cdot\gamma^\beta} - \frac{\mathcal{W}(\gamma^\beta w)}{\gamma^\beta}, \frac{\eta}{\gamma\cdot\gamma^\beta}\cdot\sum_{\rho=0}^{\alpha-1}\frac{\omega^\rho}{\gamma^\rho}\right) &\geq \mu'\left(W(\gamma^\beta w,\gamma^\beta w,\cdots,\gamma^\beta w),\eta\ \right) \\ &= \mu'\left(W(w,w,\cdots,w),\tfrac{\eta}{\omega^\beta}\right) \\ \nu\left(\frac{\mathcal{W}(\gamma^\alpha\cdot\gamma^\beta w)}{\gamma^\alpha\cdot\gamma^\beta} - \frac{\mathcal{W}(\gamma^\beta w)}{\gamma^\beta}, \frac{\eta}{\gamma\cdot\gamma^\beta}\cdot\sum_{\rho=0}^{\alpha-1}\frac{\omega^\rho}{\gamma^\rho}\right) &\geq \nu'\left(W(\gamma^\beta w,\gamma^\beta w,\cdots,\gamma^\beta w),\eta\ \right) \\ &= \nu'\left(W(w,w,\cdots,w),\tfrac{\eta}{\omega^\beta}\right) \end{aligned}\right\} \tag{29}$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$ and all $\beta, \alpha \geq 0$. Taking η by $\omega^\beta\eta$ in (29), one can reach

$$\left.\begin{aligned} \mu\left(\frac{\mathcal{W}(\gamma^\alpha\cdot\gamma^\beta w)}{\gamma^\alpha\cdot\gamma^\beta} - \frac{\mathcal{W}(\gamma^\beta w)}{\gamma^\beta}, \frac{\eta}{\gamma}\sum_{\rho=0}^{\alpha-1}\frac{\omega^\alpha\cdot\omega^\beta}{\gamma^\alpha\cdot\gamma^\beta}\right) \geq \mu'\left(W(w,w,\cdots,w),\eta\right) \\ \nu\left(\frac{\mathcal{W}(\gamma^\alpha\cdot\gamma^\beta w)}{\gamma^\alpha\cdot\gamma^\beta} - \frac{\mathcal{W}(\gamma^\beta w)}{\gamma^\beta}, \frac{\eta}{\gamma}\sum_{\rho=0}^{\alpha-1}\frac{\omega^\alpha\cdot\omega^\beta}{\gamma^\alpha\cdot\gamma^\beta}\right) \leq \nu'\left(W(w,w,\cdots,w),\eta\right) \end{aligned}\right\} \tag{30}$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$ and all $\beta, \alpha \geq 0$. The relation (30) implies that

$$\left.\begin{aligned} \mu\left(\frac{\mathcal{W}(\gamma^\alpha\cdot\gamma^\beta w)}{\gamma^\alpha\cdot\gamma^\beta} - \frac{\mathcal{W}(\gamma^\beta w)}{\gamma^\beta}, \eta\ \right) \geq \mu'\left(W(w,w,\cdots,w), \frac{\eta}{\frac{1}{\gamma}\cdot\sum_{\rho=\beta}^{\alpha-1}\frac{\omega^\rho}{\gamma^\rho}}\right) \\ \nu\left(\frac{\mathcal{W}(\gamma^\alpha\cdot\gamma^\beta w)}{\gamma^\alpha\cdot\gamma^\beta} - \frac{\mathcal{W}(\gamma^\beta w)}{\gamma^\beta}, \eta\ \right) \leq \nu'\left(W(w,w,\cdots,w), \frac{\eta}{\frac{1}{\gamma}\cdot\sum_{\rho=\beta}^{\alpha-1}\frac{\omega^\rho}{\gamma^\rho}}\right) \end{aligned}\right\} \tag{31}$$

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holds for all $w \in \mathcal{N}_1$ and all $\eta > 0$ and all $\beta, \alpha \geq 0$. Since $0 < \omega < 1$ and $\sum_{\rho=0}^{\alpha} \left(\frac{\omega}{\gamma}\right)^{\rho} < \infty$. The Cauchy criterion for convergence in IFNS shows that the sequence $\left\{\frac{\mathcal{W}(\gamma^{\alpha} w)}{\gamma^{\alpha}}\right\}$ is Cauchy in $(\mathcal{N}_2, \mu, \nu)$. Since $(\mathcal{N}_2, \mu, \nu)$ is a complete IFN-space this sequence converges to some point $\Psi(w) \in \mathcal{N}_2$. So, one can define the mapping $\Psi(w) : \mathcal{N}_1 \longrightarrow \mathcal{N}_2$ by

$$\lim_{\alpha\to\infty} \mu\left(\frac{\mathcal{W}(\gamma^{\alpha} w)}{\gamma^{\alpha}} - \Psi(w), \eta\right) = 1, \quad \lim_{\alpha\to\infty} \nu\left(\frac{\mathcal{W}(\gamma^{\alpha} w)}{\gamma^{\alpha}} - \Psi(w), \eta\right) = 0$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$. Hence

$$\frac{\mathcal{W}(\gamma^{\alpha} w)}{\gamma^{\alpha}} \xrightarrow{IF} \Psi(w), \quad as \quad n \to \infty.$$

Letting $\beta = 0$ in (31), one can attain

$$\left.\begin{aligned} \mu\left(\frac{\mathcal{W}(\gamma^{\alpha} w)}{\gamma^{\alpha}} - \mathcal{W}(w), \eta\right) &\geq \mu'\left(W(w, w, \cdots, w), \frac{\eta}{\frac{1}{\gamma} \cdot \sum_{\rho=0}^{\alpha-1} \frac{\omega^{\rho}}{\gamma^{\rho}}}\right) \\ \nu\left(\frac{\mathcal{W}(\gamma^{\alpha} w)}{\gamma^{\alpha}} - \mathcal{W}(w), \eta\right) &\leq \nu'\left(W(w, w, \cdots, w), \frac{\eta}{\frac{1}{\gamma} \cdot \sum_{\rho=0}^{\alpha-1} \frac{\omega^{\rho}}{\gamma^{\rho}}}\right) \end{aligned}\right\} \tag{32}$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$. Letting $\alpha \to \infty$ in (32), one can gain

$$\left.\begin{aligned} \mu\left(\Psi(w) - \mathcal{W}(w), \eta\right) &\geq \mu'\left(W(w, w, \cdots, w), \eta(\gamma - \omega)\right) \\ \nu\left(\Psi(w) - \mathcal{W}(w), \eta\right) &\leq \nu'\left(W(w, w, \cdots, w), \eta(\gamma - \omega)\right) \end{aligned}\right\} \tag{33}$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$. To prove $\Psi(w)$ satisfies (12), altering w_i by $\gamma^{\delta} w_i$ $(i = 1, 2, \cdots, \ell)$ in (13), one can identify

$$\left.\begin{aligned} \mu\left(\frac{1}{\gamma^{\alpha}}\left\{\mathcal{W}\left(\sum_{j=1}^{\ell} j\, \gamma^{\alpha}\, w_j\right) - \sum_{j=1}^{\ell}\left(j\, \mathcal{W}(\gamma^{\alpha}\, w_j)\right)\right\}, \eta\right) &\geq \mu'\left(W\left(\gamma^{\alpha} w_1, \gamma^{\alpha} w_2, \cdots, \gamma^{\alpha} w_{\ell}\right), \gamma^{\alpha}\, \eta\right) \\ \nu\left(\frac{1}{\gamma^{\alpha}}\left\{\mathcal{W}\left(\sum_{j=1}^{\ell} j\, \gamma^{\alpha}\, w_j\right) - \sum_{j=1}^{\ell}\left(j\, \mathcal{W}(\gamma^{\alpha}\, w_j)\right)\right\}, \eta\right) &\leq \nu'\left(W\left(\gamma^{\alpha} w_1, \gamma^{\alpha} w_2, \cdots, \gamma^{\alpha} w_{\ell}\right), \gamma^{\alpha}\, \eta\right) \end{aligned}\right\} \tag{34}$$

for all $w_1, w_2, \cdots, w_\ell \in \mathcal{N}_1$ and all $\eta > 0$. Now,

$$\mu\left(\Psi\left(\sum_{j=1}^{\ell} j\ w_j\right) - \sum_{j=1}^{\ell}\left(j\ \Psi(\ w_j)\right), \eta\right)$$
$$\geq \mu\left(\Psi\left(\sum_{j=1}^{\ell} jw_j\right) - \frac{1}{\gamma^\alpha}\mathcal{W}\left(\sum_{j=1}^{\ell} j\ \gamma^\alpha\ w_j\right), \frac{\eta}{3}\right) *$$
$$\mu\left(-\sum_{j=1}^{\ell}\left(j\ \Psi(w_j)\right) + \frac{1}{\gamma^\alpha}\sum_{j=1}^{\ell}\left(j\ \mathcal{W}(\gamma^\alpha\ w_j)\right), \frac{\eta}{3}\right) *$$
$$\mu\left(\frac{1}{\gamma^\alpha}\left\{\mathcal{W}\left(\sum_{j=1}^{\ell} j\ \gamma^\alpha\ w_j\right) - \sum_{j=1}^{\ell}\left(j\ \mathcal{W}(\gamma^\alpha\ w_j)\right)\right\}, \frac{\eta}{3}\right) \tag{35}$$

and

$$\nu\left(\Psi\left(\sum_{j=1}^{\ell} j\ w_j\right) - \sum_{j=1}^{\ell}\left(j\ \Psi(\ w_j)\right), \eta\right)$$
$$\leq \nu\left(\Psi\left(\sum_{j=1}^{\ell} jw_j\right) - \frac{1}{\gamma^\alpha}\mathcal{W}\left(\sum_{j=1}^{\ell} j\ \gamma^\alpha\ w_j\right), \frac{\eta}{3}\right) \diamond \tag{36}$$

$$\nu\left(-\sum_{j=1}^{\ell}\left(j\ \Psi(w_j)\right) + \frac{1}{\gamma^\alpha}\sum_{j=1}^{\ell}\left(j\ \mathcal{W}(\gamma^\alpha\ w_j)\right), \frac{\eta}{3}\right) \diamond$$
$$\nu\left(\frac{1}{\gamma^\alpha}\left\{\mathcal{W}\left(\sum_{j=1}^{\ell} j\ \gamma^\alpha\ w_j\right) - \sum_{j=1}^{\ell}\left(j\ \mathcal{W}(\gamma^\alpha\ w_j)\right)\right\}, \frac{\eta}{3}\right) \tag{37}$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$. Letting $\alpha \to \infty$ in (35), (36) and using (15), one can notice that $\Psi(w)$ satisfies the functional equation (12). In order to prove the existence of $\Psi(w)$ is unique, let $\Phi(w)$ be another additive functional equation satisfying (12) and (18). Hence,

$$\begin{aligned}
\mu(\Psi(w)-\Phi(w),\eta) &\geq \mu\left(\Psi(\gamma^\alpha w)-\mathcal{W}(\gamma^\alpha w),\frac{\eta\cdot\gamma^\alpha}{2}\right) * \mu\left(\mathcal{W}(\gamma^\alpha w)-\Phi(\gamma^\alpha w),\frac{\eta\cdot\gamma^\alpha}{2}\right)\\
&\geq \mu'\left(W(\gamma^\alpha w,\gamma^\alpha w,\cdots,\gamma^\alpha w),\frac{\eta\ \gamma^\alpha|\gamma-\omega|}{2}\right)\\
&= \mu'\left(W(w,w,\cdots,w),\frac{\eta\gamma^\alpha|\gamma-\omega|}{2\cdot\omega^\alpha}\right)\\
\nu(\Psi(w)-\Phi(w),\eta) &\leq \nu\left(\Psi(\gamma^\alpha w)-\mathcal{W}(\gamma^\alpha w),\frac{\eta\cdot\gamma^\alpha}{2}\right) \diamond \nu\left(\mathcal{W}(\gamma^\alpha w)-\Phi(\gamma^\alpha w),\frac{\eta\cdot\gamma^\alpha}{2}\right)\\
&\leq \nu'\left(W(\gamma^\alpha w,\gamma^\alpha w,\cdots,\gamma^\alpha w),\frac{\eta\ \gamma^\alpha|\gamma-\omega|}{2}\right)\\
&= \nu'\left(W(w,w,\cdots,w),\frac{\eta\gamma^\alpha|\gamma-\omega|}{2\cdot\omega^\alpha}\right)
\end{aligned}$$

for all $w\in\mathcal{N}_1$ and all $\eta>0$. In view of $\lim_{\alpha\to\infty}\frac{\eta\ \gamma^\alpha|\gamma-\omega|}{2\omega^\alpha}=\infty$, one can acquire that

$$\left.\begin{aligned}
\lim_{\alpha\to\infty}\mu'\left(W(w,w,\cdots,w),\tfrac{\eta\ \gamma^\alpha|\gamma-\omega|}{2\omega^\alpha}\right)&=1\\
\lim_{\alpha\to\infty}\nu'\left(W(w,w,\cdots,w),\tfrac{\eta\ \gamma^\alpha|\gamma-\omega|}{2\omega^\alpha}\right)&=0
\end{aligned}\right\}$$

for all $w\in\mathcal{N}_1$ and all $\eta>0$. Thus

$$\left.\begin{aligned}
\mu(\Psi(w)-\Phi(w),\eta)&=1\\
\nu(\Psi(w)-\Phi(w),\eta)&=0
\end{aligned}\right\}$$

for all $w\in\mathcal{N}_1$ and all $\eta>0$. Hence, $\Psi(w)=\Phi(w)$. Therefore, $\Psi(w)$ is unique. So, the theorem holds for $\beta=1$.

Case 2: Taking w as $\frac{w}{\gamma}$ in (20), one can see

$$\left.\begin{aligned}
\mu\left(\mathcal{W}(w)-\gamma\mathcal{W}\left(\tfrac{w}{\gamma}\right),\eta\right)&\geq\mu'\left(W\left(\tfrac{w}{\gamma},\tfrac{w}{\gamma}\cdots,\tfrac{w}{\gamma}\right),\eta\right)\\
\nu\left(\mathcal{W}(w)-\gamma\mathcal{W}\left(\tfrac{w}{\gamma}\right),\eta\right)&\leq\nu'\left(W\left(\tfrac{w}{\gamma},\tfrac{w}{\gamma},\cdots,\tfrac{w}{\gamma}\right),\eta\right)
\end{aligned}\right\}\quad(38)$$

for all $w\in\mathcal{N}_1$ and all $\eta>0$. The rest of the proof is identical to that of Case 1. This completes the proof of the theorem.

The resulting corollary is an immediate consequence of Theorem 3.1, regarding the stability of the functional equation (12).

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Corollary 3.2 If $\mathcal{W} : \mathcal{N}_1 \to \mathcal{N}_2$ is a function satisfying the inequalities

$$\left.\begin{array}{l} \mu\left(\mathcal{W}\left(\sum_{j=1}^{\ell}\ j\ w_j\right) - \sum_{j=1}^{\ell}\left(j\ \mathcal{W}(w_j)\right), \eta\ \right) \geq \left\{\begin{array}{l} \mu'\left(a,\eta\right), \\ \mu'\left(a\sum_{j=1}^{\ell}|w_j|^b,\eta\right), \\ \mu'\left(\sum_{j=1}^{\ell}a_j|w_j|^{b_j},\eta\right), \\ \mu'\left(a\prod_{j=1}^{\ell}|w_j|^b,\eta\right), \\ \mu'\left(\prod_{j=1}^{\ell}a_j|w_j|^{b_j},\eta\right), \end{array}\right. \\ \nu\left(\mathcal{W}\left(\sum_{j=1}^{\ell}\ j\ w_j\right) - \sum_{j=1}^{\ell}\left(j\ \mathcal{W}(w_j)\right), \eta\ \right) \leq \left\{\begin{array}{l} \nu'\left(a,\eta\right), \\ \nu'\left(a\sum_{j=1}^{\ell}|w_j|^b,\eta\right), \\ \nu'\left(\sum_{j=1}^{\ell}a_j|w_j|^{b_j},\eta\right), \\ \nu'\left(a\prod_{j=1}^{\ell}|w_j|^b,\eta\right), \\ \nu'\left(\prod_{j=1}^{\ell}a_j|w_j|^{b_j},\eta\right), \end{array}\right. \end{array}\right\} \tag{39}$$

for all $w_1, w_2. \cdots, w_\ell \in \mathcal{N}_1$ and all $\eta > 0$, with positive numbers $a > 0$ and $b, b_j, \ell b, \sum_{j=1}^{\ell} b_j \neq 1$. Then there exists a unique additive mapping $\Psi(w) : \mathcal{N}_1 \longrightarrow \mathcal{N}_2$ satisfying the functional equation (12) and

$$\left.\begin{array}{l} \mu\left(\mathcal{W}(w) - \Psi(w), \eta\ \right) \geq \left\{\begin{array}{l} \mu'\left(a, |\gamma-1|\eta\right), \\ \mu'\left(a\ \ell\ |w|^b, |\gamma-\gamma^b|\eta\right), \\ \mu'\left(\sum_{j=1}^{\ell}a_j\ |w|^{b_j}, \eta\ \sum_{j=1}^{\ell}|\gamma-\gamma^{b_j}|\right), \\ \mu'\left(a\ |w|^{\ell b}, |\gamma-\gamma^{\ell b}|\eta\right), \\ \mu'\left(\prod_{j=1}^{\ell}a_j\ |w|^{\sum_{j=1}^{\ell}b_j}, \eta\ |\gamma-\gamma^{\sum_{j=1}^{\ell}b_j}|\right), \end{array}\right. \\ \nu\left(\mathcal{W}(w) - \Psi(w), \eta\ \right) \leq \left\{\begin{array}{l} \nu'\left(a, |\gamma-1|\eta\right), \\ \nu'\left(a\ \ell\ |w|^b, |\gamma-\gamma^b|\eta\right), \\ \nu'\left(\sum_{j=1}^{\ell}a_j\ |w|^{b_j}, \eta\ \sum_{j=1}^{\ell}|\gamma-\gamma^{b_j}|\right), \\ \nu'\left(a\ |w|^{\ell b}, |\gamma-\gamma^{\ell b}|\eta\right), \\ \nu'\left(\prod_{j=1}^{\ell}a_j\ |w|^{\sum_{j=1}^{\ell}b_j}, \eta\ |\gamma-\gamma^{\sum_{j=1}^{\ell}b_j}|\right), \end{array}\right. \end{array}\right\} \tag{40}$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$.

Radus Method of (12)

Theorem 3.3 If $\mathcal{W} : \mathcal{N}_1 \to \mathcal{N}_2$ is a function satisfying the inequality (13). Assume

$$\lambda_d = \begin{cases} \gamma & d = 0; \\ \frac{1}{\gamma} & d = 1. \end{cases} \tag{41}$$

and $W : \mathcal{N}_1^\ell \to [0, \infty)$ be a function with the condition

$$\left.\begin{array}{l} \lim\limits_{\alpha\to\infty} \mu' \left(W\left(\lambda_d^\alpha w_1, \lambda_d^\alpha w_2, \cdots, \lambda_d^\alpha w_\ell\right), \lambda_d^\alpha \eta\right) = 1 \\ \lim\limits_{\alpha\to\infty} \nu' \left(W\left(\lambda_d^\alpha w_1, \lambda_d^\alpha w_2, \cdots, \lambda_d^\alpha w_\ell\right), \lambda_d^\alpha \eta\right) = 0 \end{array}\right\} \tag{42}$$

for all $w_1, w_2, \cdots, w_\ell \in \mathcal{N}_1$ and all $\eta > 0$. If there exists $L = L(d)$ such that the functions

$$\left.\begin{array}{l} \mu' \left(W(w, w, \cdots, w), \eta\right) = \mu' \left(W\left(\frac{w}{\gamma}, \frac{w}{\gamma}, \cdots, \frac{w}{\gamma}\right), \eta\right); \\ \nu' \left(W(w, w, \cdots, w), \eta\right) = \nu' \left(W\left(\frac{w}{\gamma}, \frac{w}{\gamma}, \cdots, \frac{w}{\gamma}\right), \eta\right) \end{array}\right\} \tag{43}$$

has the property

$$\left.\begin{array}{l} \mu' \left(\frac{1}{\lambda_d} W\left(\lambda_d w, \lambda_d w, \cdots, \lambda_d w\right), \eta\right) = \mu' \left(L\, W(w, w, \cdots, w), \eta\right) \\ \nu' \left(\frac{1}{\lambda_d} W\left(\lambda_d w, \lambda_d w, \cdots, \lambda_d w\right), \eta\right) = \nu' \left(L\, W(w, w, \cdots, w), \eta\right) \end{array}\right\} \tag{44}$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$. Then there exists a unique additive mapping $\Psi(w) : \mathcal{N}_1 \to \mathcal{N}_2$ satisfying the functional equation (12) and

$$\left.\begin{array}{l} \mu \left(\mathcal{W}(w) - \Psi(w), \eta\right) \geq \mu' \left(\frac{L^{1-d}}{1-L} W(w, w, \cdots, w), \eta\right) \\ \nu \left(\mathcal{W}(w) - \Psi(w), \eta\right) \leq \nu' \left(\frac{L^{1-d}}{1-L} W(w, w, \cdots, w), \eta\right) \end{array}\right\} \tag{45}$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$.

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Proof. Define a set $\Lambda = \{\mathcal{X}/\mathcal{X} : \mathcal{N}_1 \to \mathcal{N}_2,\ \mathcal{X}(0) = 0\}$ and introduce the generalized metric on the Λ as

$$D(\mathcal{X},\mathcal{Y}) = \inf\left\{L \in (0,\infty) : \left\{\begin{array}{l} \mu(\mathcal{X}(w)-\mathcal{Y}(w),\eta) \geq \mu'(L\ W(w,w,\cdots,w),\eta), \\ \nu(\mathcal{X}(w)-\mathcal{Y}(w),\eta) \leq \nu'(L\ W(w,w,\cdots,w),\eta), \end{array}\right\}\right\} \tag{46}$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$. It is easy to see that (Λ, D) is complete. Also, define a function $T : \Lambda \to \Lambda$ by

$$T\mathcal{X}(w) = \frac{1}{\lambda_d}\mathcal{X}(\lambda_d w), \quad \text{for all} \quad w \in \mathcal{N}_1. \tag{47}$$

Now, from (46) and $\mathcal{X},\mathcal{Y} \in \Lambda$, for all $w \in \mathcal{N}_1$ and all $\eta > 0$, one can land

$$\left\{\begin{array}{l} \left\{\begin{array}{l} \inf\{1 \in (0,\infty) : \mu(\mathcal{X}(w)-\mathcal{Y}(w),\eta) \geq \mu'(W(w,w,\cdots,w),\eta),\} \\ \inf\{1 \in (0,\infty) : \mu(\frac{1}{\lambda_d}\mathcal{X}(\lambda_d w)-\frac{1}{\lambda_d}\mathcal{Y}(\lambda_d w),\eta) \geq \mu'(W(\lambda_d w,\lambda_d w,\cdots,\lambda_d w),\lambda_d\eta),\} \\ \inf\{L \in (0,\infty) : \mu(\frac{1}{\lambda_d}\mathcal{X}(\lambda_d w)-\frac{1}{\lambda_d}\mathcal{Y}(\lambda_d w),\eta) \geq \mu'(L\ W(w,w,\cdots,w),\eta),\} \\ \inf\{L \in (0,\infty) : \mu(T\mathcal{X}(w)-T\mathcal{Y}(w),\eta) \geq \mu'(L\ W(w,w,\cdots,w),\eta),\} \end{array}\right. \\ \\ \left\{\begin{array}{l} \inf\{1 \in (0,\infty) : \nu(\mathcal{X}(w)-\mathcal{Y}(w),\eta) \leq \nu'(W(w,w,\cdots,w),\eta),\} \\ \inf\{1 \in (0,\infty) : \nu(\frac{1}{\lambda_d}\mathcal{X}(\lambda_d w)-\frac{1}{\lambda_d}\mathcal{Y}(\lambda_d w),\eta) \leq \nu'(W(\lambda_d w,\lambda_d w,\cdots,\lambda_d w),\lambda_d\eta),\} \\ \inf\{L \in (0,\infty) : \nu(\frac{1}{\lambda_d}\mathcal{X}(\lambda_d w)-\frac{1}{\lambda_d}\mathcal{Y}(\lambda_d w),\eta) \leq \nu'(L\ W(w,w,\cdots,w),\eta),\} \\ \inf\{L \in (0,\infty) : \nu(T\mathcal{X}(w)-T\mathcal{Y}(w),\eta) \leq \nu'(L\ W(w,w,\cdots,w),\eta),\} \end{array}\right. \end{array}\right\}$$

This implies $D(T\mathcal{X},T\mathcal{Y}) \leq LD(\mathcal{X},\mathcal{Y})$, for all $\mathcal{X},\mathcal{Y} \in \Lambda$. i.e., T is a strictly contractive mapping on Λ with Lipschitz constant L .

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With the help of (IFN4), (IFN10), (44) and (47) for the case $d = 0$, (21) reduces to

$$\left.\begin{array}{l} \mu\left(\dfrac{\mathcal{W}(\gamma w)}{\gamma} - \mathcal{W}(w), \eta\right) \geq \mu'\left(\dfrac{1}{\gamma} W\left(w, w, \cdots, w\right), \eta\right) \\ \Longrightarrow \mu\Big(T\mathcal{W}(w) - \mathcal{W}(w), \eta\Big) \geq \mu'\left(L^1 W\left(w, w, \cdots, w\right), \eta\right) = \mu'\left(L^{1-d}\, W\left(w, w, \cdots, w\right), \eta\right) \\ \nu\left(\dfrac{\mathcal{W}(\gamma w)}{\gamma} - \mathcal{W}(w), \eta\right) \leq \nu'\left(\dfrac{1}{\gamma} W\left(w, w, \cdots, w\right), \eta\right) \\ \Longrightarrow \nu\Big(T\mathcal{W}(w) - \mathcal{W}(w), \eta\Big) \leq \nu'\left(L^1 W\left(w, w, \cdots, w\right), \eta\right) = \nu'\left(L^{1-d}\, W\left(w, w, \cdots, w\right), \eta\right) \end{array}\right\} \tag{48}$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$.
With the help of (IFN4), (IFN10), (44) and (47) for the case $d = 1$, (38) reduces to

$$\left.\begin{array}{l} \mu\left(\mathcal{W}(w) - \gamma\mathcal{W}\left(\dfrac{w}{\gamma}\right), \eta\right) \geq \mu'\left(W\left(\dfrac{w}{\gamma}, \dfrac{w}{\gamma}, \cdots \dfrac{w}{\gamma}\right), \eta\right) \\ \Longrightarrow \mu\Big(\mathcal{W}(w) - T\mathcal{W}(w), \eta\Big) \geq \mu'\left(L^0\, W\left(w, w, \cdots, w\right), \eta\right) = \mu'\left(L^{1-d}\, W\left(w, w, \cdots, w\right), \eta\right) \\ \nu\left(\mathcal{W}(w) - \gamma\mathcal{W}\left(\dfrac{w}{\gamma}\right), \eta\right) \leq \nu'\left(W\left(\dfrac{w}{\gamma}, \dfrac{w}{\gamma}, \cdots \dfrac{w}{\gamma}\right), \eta\right) \\ \Longrightarrow \nu\Big(\mathcal{W}(w) - T\mathcal{W}(w), \eta\Big) \leq \nu'\left(L^0\, W\left(w, w, \cdots, w\right), \eta\right) = \nu'\left(L^{1-d}\, W\left(w, w, \cdots, w\right), \eta\right) \end{array}\right\} \tag{49}$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$. Combining the above two cases, we arrive

$$\inf\left\{L^{1-d} \in (0, \infty) : \left\{\begin{array}{l} \mu(T\ \mathcal{W}(w) - \mathcal{W}(w), \eta) \geq \mu'(L^{1-d} W\left(w, w, \cdots, w\right), \eta), \\ \nu(T\ \mathcal{W}(w) - \mathcal{W}(w), \eta) \leq \nu'(L^{1-d} W\left(w, w, \cdots, w\right), \eta), \end{array}\right\}\right\} \tag{50}$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$. Hence property (FPC1) of Theorem 1.1 holds. By (FPC2) of Theorem 1.1, it follows that there exists a fixed point Ψ of T in Λ such that

$$\left.\begin{array}{l} \lim\limits_{\alpha\to\infty} \mu\left(\frac{\mathcal{W}(\lambda_d^\alpha w)}{\lambda_d^\alpha} - \Psi(w), \eta\right) = 1, \\ \lim\limits_{\alpha\to\infty} \nu\left(\frac{\mathcal{W}(\lambda_d^\alpha w)}{\lambda_d^\alpha} - \Psi(w), \eta\right) = 0 \end{array}\right\}$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$. To order to prove $\Psi : \mathcal{N}_1 \longrightarrow \mathcal{N}_2$ is additive, the proof is similar to that of Theorem 3.1. By (FPC3) of Theorem 1.1, Ψ is the unique fixed point of T in the set $\Omega = \{\Psi \in \Lambda : d(\mathcal{W}, \Psi) < \infty\}$ and

$$\left.\begin{array}{l}\mu(\mathcal{W}(w)-\Psi(w),\eta)\geq\mu'(L^{1-d}W\left(w,w,\cdots,w\right),\eta),\\ \nu(\mathcal{W}(w)-\Psi(w),\eta)\leq\nu'(L^{1-d}W\left(w,w,\cdots,w\right),\eta),\end{array}\right\}$$

for all $w\in\mathcal{N}_1$ and and all $\eta>0$. Finally by (FPC4) of Theorem 1.1, we obtain

$$\left.\begin{array}{l}\mu\left(\mathcal{W}(w)-\Psi(w),\eta\right)\geq\mu'\left(\frac{L^{1-d}}{1-L}W\left(w,w,\cdots,w\right),\eta\right)\\ \nu\left(\mathcal{W}(w)-\Psi(w),\eta\right)\leq\nu'\left(\frac{L^{1-d}}{1-L}W\left(w,w,\cdots,w\right),\eta\right)\end{array}\right\}$$

for all $w\in\mathcal{N}_1$ and all $\eta>0$. This complets the proof ofthe theorem.

The following corollary is an immediate consequence of Theorem 3.3 concerning the some stabilities of (12).

Corollary 3.4 If $\mathcal{W}:\mathcal{N}_1\to\mathcal{N}_2$ is a function satisfying the inequalities

$$\left.\begin{array}{l}\mu\left(\mathcal{W}\left(\sum_{j=1}^{\ell}\ j\ w_j\right)-\sum_{j=1}^{\ell}\left(j\ \mathcal{W}(w_j)\right),\eta\ \right)\geq\left\{\begin{array}{l}\mu'\left(a,\eta\right),\\ \mu'\left(a\sum_{j=1}^{\ell}|w_j|^b,\eta\right),\\ \mu'\left(a\prod_{j=1}^{\ell}|w_j|^b,\eta\right),\end{array}\right.\\ \nu\left(\mathcal{W}\left(\sum_{j=1}^{\ell}\ j\ w_j\right)-\sum_{j=1}^{\ell}\left(j\ \mathcal{W}(w_j)\right),\eta\ \right)\leq\left\{\begin{array}{l}\nu'\left(a,\eta\right),\\ \nu'\left(a\sum_{j=1}^{\ell}|w_j|^b,\eta\right),\\ \nu'\left(a\prod_{j=1}^{\ell}|w_j|^b,\eta\right),\end{array}\right.\end{array}\right\}\quad(51)$$

for all $w_1,w_2.\cdots,w_\ell\in\mathcal{N}_1$ and all $\eta>0$, with positive numbers $a>0$ and $b,\ell b\neq 1$. Then there exists a unique additive mapping $\Psi(w):\mathcal{N}_1\longrightarrow\mathcal{N}_2$ satisfying the functional equation (12) and

$$\left.\begin{array}{l}\mu\left(W(w)-\Psi(w),\eta\ \right)\geq\left\{\begin{array}{l}\mu'\left(a,|\gamma-1|\eta\right),\\ \mu'\left(a\ \ell\ |w|^b,|\gamma-\gamma^b|\eta\right),\\ \mu'\left(a\ |w|^{\ell b},|\gamma-\gamma^{\ell b}|\eta\right),\end{array}\right.\\ \nu\left(W(w)-\Psi(w),\eta\ \right)\leq\left\{\begin{array}{l}\nu'\left(a,|\gamma-1|\eta\right),\\ \nu'\left(a\ \ell\ |w|^b,|\gamma-\gamma^b|\eta\right),\\ \nu'\left(a\ |w|^{\ell b},|\gamma-\gamma^{\ell b}|\eta\right),\end{array}\right.\end{array}\right\}\quad(52)$$

for all $w\in\mathcal{N}_1$ and all $\eta>0$.

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Proof. It follows from LHS of (51) and Theorem 3.3 of (13) by replacing w_j by $\lambda_d^\alpha w_j$, one can notice that

$$\left.\begin{array}{l} \mu'\left(\frac{1}{\lambda_d^\alpha}W(\lambda_d^\alpha w_1, \lambda_d^\alpha w_2, \cdots, \lambda_d^\alpha w_\ell), \lambda_d^\alpha \eta\right) \geq \begin{cases} \mu'\left(a, \lambda_d^\alpha \eta\right), \\ \mu'\left(a \sum_{j=1}^{\ell} |\lambda_d^\alpha w_j|^b, \lambda_d^\alpha \eta\right), \\ \mu'\left(a \prod_{j=1}^{\ell} |\lambda_d^\alpha w_j|^b, \lambda_d^\alpha \eta\right), \end{cases} \\ \nu'\left(\frac{1}{\lambda_d^\alpha}W(\lambda_d^\alpha w_1, \lambda_d^\alpha w_2, \cdots, \lambda_d^\alpha w_\ell), \lambda_d^\alpha \eta\right) \leq \begin{cases} \nu'\left(a, \lambda_d^\alpha \eta\right), \\ \nu'\left(a \sum_{j=1}^{\ell} |\lambda_d^\alpha w_j|^b, \lambda_d^\alpha \eta\right), \\ \nu'\left(a \prod_{j=1}^{\ell} |\lambda_d^\alpha w_j|^b, \lambda_d^\alpha \eta\right), \end{cases} \end{array}\right\} \quad (53)$$

for all $w_1, w_2. \cdots, w_\ell \in \mathcal{N}_1$ and all $\eta > 0$ and letting α tends to infinity in (53) one can view that (42) holds.

From (43), (53) and (44), one can arrive

$$\mu'\left(W(w, w, \cdots, w), \eta\right) = \mu'\left(W\left(\frac{w}{\gamma}, \frac{w}{\gamma}, \cdots \frac{w}{\gamma}\right), \eta\right) = \begin{cases} \mu'\left(a, \eta\right), \\ \mu'\left(\dfrac{a\ell|w|^b}{\gamma^b}, \eta\right), \\ \mu'\left(\dfrac{a|w|^{\ell b}}{\gamma^{\ell b}}, \eta\right), \end{cases}$$

$$\nu'\left(W(w, w, \cdots, w), \eta\right) = \nu'\left(W\left(\frac{w}{\gamma}, \frac{w}{\gamma}, \cdots \frac{w}{\gamma}\right), \eta\right) = \begin{cases} \nu'\left(a, \eta\right), \\ \nu'\left(\dfrac{a\ell|w|^b}{\gamma^b}, \eta\right), \\ \nu'\left(\dfrac{a|w|^{\ell b}}{\gamma^{\ell b}}, \eta\right), \end{cases}$$

$$\mu'\left(\frac{1}{\lambda_d}W\left(\lambda_d w, \lambda_d w, \cdots, \lambda_d w\right), \eta\right) = \mu'\left(L\ W(w, w, \cdots, w), \eta\right) = \begin{cases} \mu'\left(\lambda_d^{-1} a, \eta\right), \\ \mu'\left(\lambda_d^{b-1}\ a||w||^b, \eta\right), \\ \mu'\left(\lambda_d^{\ell b-1}\ a||w||^{\ell b}, \eta\right), \end{cases}$$

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$$\nu'\left(\frac{1}{\lambda_d}W\left(\lambda_d w, \lambda_d w, \cdots, \lambda_d w\right), \eta\right) = \nu'\left(L\ W(w, w, \cdots, w), \eta\right) = \begin{cases} \nu'\left(\lambda_d^{-1} a, \eta\right), \\ \nu'\left(\lambda_d^{b-1}\ a||w||^b, \eta\right), \\ \nu'\left(\lambda_d^{\ell b-1}\ a||w||^{\ell b}, \eta\right), \end{cases}$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$.

$$For \quad d = 0; \quad L = \lambda_d^{-1} = \gamma^{-1}$$

$$\left.\begin{aligned} \mu\left(\mathcal{W}(w) - \Psi(w), \eta\right) &\geq \mu'\left(\tfrac{(\gamma^{-1})^{1-0}}{1-\gamma^{-1}}\ W(w, w, \cdots, w), \eta\right) = \mu'\left(a, (\gamma - 1)\eta\right) \\ \nu\left(\mathcal{W}(w) - \Psi(w), \eta\right) &\leq \nu'\left(\tfrac{(\gamma^{-1})^{1-0}}{1-\gamma^{-1}}\ W(w, w, \cdots, w), \eta\right) = \nu'\left(a, (\gamma - 1)\eta\right) \end{aligned}\right\}$$

$$For \quad d = 1; \qquad L = \lambda_d^{-1} = \frac{1}{\gamma^{-1}} = \gamma$$

$$\left.\begin{aligned} \mu\left(\mathcal{W}(w) - \Psi(w), \eta\right) &\geq \mu'\left(\tfrac{(\gamma)^{1-1}}{1-\gamma}\ W(w, w, \cdots, w), \eta\right) = \mu'\left(a, (1 - \gamma)\eta\right) \\ \nu\left(\mathcal{W}(w) - \Psi(w), \eta\right) &\leq \nu'\left(\tfrac{(\gamma)^{1-1}}{1-\gamma}\ W(w, w, \cdots, w), \eta\right) = \nu'\left(a, (1 - \gamma)\eta\right) \end{aligned}\right\}$$

$$For \quad d = 0; \quad L = \lambda_d^{b-1} = \gamma^{b-1}$$

$$\left.\begin{aligned} \mu\left(\mathcal{W}(w) - \Psi(w), \eta\right) &\geq \mu'\left(\tfrac{(\gamma^{b-1})^{1-0}}{1-\gamma^{b-1}}\ W(w, w, \cdots, w), \eta\right) = \mu'\left(a\ \ell\ ||w||^b, (\gamma - \gamma^b)\eta\right) \\ \nu\left(\mathcal{W}(w) - \Psi(w), \eta\right) &\leq \nu'\left(\tfrac{(\gamma^{b-1})^{1-0}}{1-\gamma^{b-1}}\ W(w, w, \cdots, w), \eta\right) = \nu'\left(a\ \ell\ ||w||^b, (\gamma - \gamma^b)\eta\right) \end{aligned}\right\}$$

$$For \quad d = 1; \quad L = \lambda_d^{b-1} = \frac{1}{\gamma^{b-1}} = \gamma^{1-b}$$

$$\left.\begin{aligned} \mu\left(\mathcal{W}(w) - \Psi(w), \eta\right) &\geq \mu'\left(\tfrac{(\gamma^{1-b})^{1-1}}{1-\gamma^{1-b}}\ W(w, w, \cdots, w), \eta\right) = \mu'\left(a\ \ell\ ||w||^b, (\gamma^b - \gamma)\eta\right) \\ \nu\left(\mathcal{W}(w) - \Psi(w), \eta\right) &\leq \nu'\left(\tfrac{(\gamma^{1-b})^{1-1}}{1-\gamma^{1-b}}\ W(w, w, \cdots, w), \eta\right) = \nu'\left(a\ \ell\ ||w||^b, (\gamma^b - \gamma)\eta\right) \end{aligned}\right\}$$

$$For \quad d = 0; \quad L = \lambda_d^{b-1} = \gamma^{\ell b-1}$$

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$$\left.\begin{array}{l}\mu\left(\mathcal{W}(w)-\Psi(w),\eta\right)\geq\mu'\left(\frac{(\gamma^{\ell b-1})^{1-0}}{1-\gamma^{\ell b-1}}\;W(w,w,\cdots,w),\eta\right)=\mu'\left(a\;||w||^{\ell b},(\gamma-\gamma^{\ell b})\eta\right)\\ \nu\left(\mathcal{W}(w)-\Psi(w),\eta\right)\leq\nu'\left(\frac{(\gamma^{\ell b-1})^{1-0}}{1-\gamma^{\ell b-1}}\;W(w,w,\cdots,w),\eta\right)=\nu'\left(a\;||w||^{\ell b},(\gamma-\gamma^{\ell b})\eta\right)\end{array}\right\}$$

$$For\quad d=1;\quad L=\lambda_d^{b-1}=\frac{1}{\gamma^{\ell b-1}}=\gamma^{1-\ell b}$$

$$\left.\begin{array}{l}\mu\left(\mathcal{W}(w)-\Psi(w),\eta\right)\geq\mu'\left(\frac{(\gamma^{1-\ell b})^{1-1}}{1-\gamma^{1-\ell b}}\;W(w,w,\cdots,w),\eta\right)=\mu'\left(a\;||w||^{\ell b},(\gamma^{\ell b}-\gamma)\eta\right)\\ \nu\left(\mathcal{W}(w)-\Psi(w),\eta\right)\leq\nu'\left(\frac{(\gamma^{1-\ell b})^{1-1}}{1-\gamma^{1-\ell b}}\;W(w,w,\cdots,w),\eta\right)=\nu'\left(a\;||w||^{\ell b},(\gamma^{\ell b}-\gamma)\eta\right)\end{array}\right\}$$

Hence the proof is complete.

Stability In Intuitionistic Fuzzy Banach Algebra

In this section, the generalized Ulam - Hyers stability of the functional equation (12) in Intuitionistic Fuzzy Banach Algebra is established.

Definitions and Notations of Intuitionistic Fuzzy Banach Algebra

Definition 4.1 A Intuitionistic Fuzzy Banach space A is said to be a Intuitionistic Fuzzy Banach algebra if it satisfies the condition

(IFN14) $\mu(x,t)*\mu(y,s)\leq\mu(xy,ts)$,
(IFN15) $\nu(x,t)\diamond\nu(y,s)\geq\nu(xy,ts)$,

for all $x,y\in A$.

Definition 4.2 Let A and B be real Banach algebras. A mapping $H:A\to B$ is called a Intuitionistic Fuzzy Algebra Homomorphism if

$$H(xy)=H(x)H(y)\quad\text{for all}\quad x,y\;\in\;A.$$

Also, in general

$$H\left(\prod_i^n x_i\right)=\prod_i^n H(x_i)\quad\text{for all}\quad x_i\;\in\;A.$$

Definition 4.3 Let A and B be real Banach algebras. A mapping $D:A\to B$ is called a Intuitionistic Fuzzy Algebra Derivation if

$$D(xy)=D(x)y+xD(y)\quad\text{for all}\quad x,y\;\in\;A.$$

Also, in general

$$D\left(\prod_{i}^{n} x_i\right) = \prod_{i=1}^{n} x_i \; D(x_j); \quad i < j < n \quad i \neq j \quad \text{for all} \quad x_i \in A.$$

To prove stability results, let us take $\mathcal{M}_1$ be an Intuitionistic Fuzzy normed algebra and $\mathcal{M}_2$ be an Intuitionistic Fuzzy Banach algebra .

Homomorphism Stability Result: Hyers and Radus Method

Theorem 4.4 If $\mathcal{W} : \mathcal{M}_1 \to \mathcal{M}_2$ is a function satisfying the inequalities (13) and

$$\left.\begin{aligned} \mu\left(\Psi\left(\prod_j^\ell w_j\right) - \prod_j^\ell \Psi(w_j), \eta\right) \geq \mu'\left(W\left(w_1, w_2, \cdots, w_\ell\right), \eta\right) \\ \nu\left(\Psi\left(\prod_j^\ell w_j\right) - \prod_j^\ell \Psi(w_j), \eta\right) \leq \nu'\left(W\left(w_1, w_2, \cdots, w_\ell\right), \eta\right) \end{aligned}\right\} \tag{54}$$

where $\mathcal{W} : \mathcal{M}_1^\ell \longrightarrow (0, 1]$ with the conditions (14), (15) and

$$\left.\begin{aligned} \lim_{\alpha\to\infty} \mu'\left(W\left(\gamma^{\ell\alpha\beta}w_1, \gamma^{\ell\alpha\beta}w_2, \cdots, \gamma^{\ell\alpha\beta}w_\ell\right), \gamma^{\ell\alpha\beta}\eta\right) = 1 \\ \lim_{\alpha\to\infty} \nu'\left(W\left(\gamma^{\ell\alpha\beta}w_1, \gamma^{\ell\alpha\beta}w_2, \cdots, \gamma^{\ell\alpha\beta}w_\ell\right), \gamma^{\ell\alpha\beta}\eta\right) = 0 \end{aligned}\right\} \tag{55}$$

for all $w_1, w_2, \cdots, w_\ell \in \mathcal{M}_1$ and all $\eta > 0$ with (16) holds. Then there exists a unique homomorphism mapping $\Psi(w) : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ given in (17) and satisfying the functional equation (12) and (18) for all $w \in \mathcal{M}_1$ and all $\eta > 0$.

Proof. In view of Theorem 3.1, there exists a unique additive mapping $\Psi(w) : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ given in (17) and satisfying the functional equation (12) and (18) for all $w \in \mathcal{M}_1$ and all $\eta > 0$. It follows from (54) and (55), one can obtain

$$\left.\begin{aligned} \mu\left(\Psi\left(\prod_j^\ell w_j\right) - \prod_j^\ell \Psi(w_j), \eta\right) &= \mu\left(\frac{1}{\gamma^{\ell\alpha}}\left(\mathcal{W}\left(\prod_j^\ell \gamma^{\ell\alpha}w_j\right) - \prod_j^\ell \mathcal{W}(\gamma^{\ell\alpha}w_j)\right), \gamma^{\ell\alpha}\eta\right) \\ &\geq \mu'\left(W\left(\gamma^{\ell\alpha}w_1, \gamma^{\ell\alpha}w_2, \cdots, \gamma^{\ell\alpha}w_\ell\right)\gamma^{\ell\alpha}\eta\right) \\ &\to \quad 1 \quad \text{as} \quad \alpha \to \infty \\ \nu\left(\Psi\left(\prod_j^\ell w_j\right) - \prod_j^\ell \Psi(w_j), \eta\right) &= \nu\left(\frac{1}{\gamma^{\ell\alpha}}\left(\mathcal{W}\left(\prod_j^\ell \gamma^{\ell\alpha}w_j\right) - \prod_j^\ell \mathcal{W}(\gamma^{\ell\alpha}w_j)\right), \gamma^{\ell\alpha}\eta\right) \\ &\leq \nu'\left(W\left(\gamma^{\ell\alpha}w_1, \gamma^{\ell\alpha}w_2, \cdots, \gamma^{\ell\alpha}w_\ell\right)\gamma^{\ell\alpha}\eta\right) \\ &\to \quad 0 \quad \text{as} \quad \alpha \to \infty \end{aligned}\right\}$$

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for all $w \in \mathcal{M}_1$ and all $\eta > 0$. Thus $\Psi(w)$ is a unique homomorphism mapping.

Corollary 4.5 If $\mathcal{W} : \mathcal{M}_1 \to \mathcal{M}_2$ is a function satisfying the inequalities (39) and

$$\left.\begin{array}{l} \mu\left(\Psi\left(\prod_j^\ell w_j\right) - \prod_j^\ell \Psi(w_j), \eta\right) \geq \begin{cases} \mu'(a,\eta), \\ \mu'\left(a\sum_{j=1}^\ell |w_j|^b, \eta\right), \\ \mu'\left(\sum_{j=1}^\ell a_j|w_j|^{b_j}, \eta\right), \\ \mu'\left(a\prod_{j=1}^\ell |w_j|^b, \eta\right), \\ \mu'\left(\prod_{j=1}^\ell a_j|w_j|^{b_j}, \eta\right), \end{cases} \\ \nu\left(\Psi\left(\prod_j^\ell w_j\right) - \prod_j^\ell \Psi(w_j), \eta\right) \leq \begin{cases} \nu'(a,\eta), \\ \nu'\left(a\sum_{j=1}^\ell |w_j|^b, \eta\right), \\ \nu'\left(\sum_{j=1}^\ell a_j|w_j|^{b_j}, \eta\right), \\ \nu'\left(a\prod_{j=1}^\ell |w_j|^b, \eta\right), \\ \nu'\left(\prod_{j=1}^\ell a_j|w_j|^{b_j}, \eta\right), \end{cases} \end{array}\right\} \tag{56}$$

for all $w_1, w_2.\cdots, w_\ell \in \mathcal{M}_1$ and all $\eta > 0$, with positive numbers $a > 0$ and $b, b_j, \ell b\sum_{j=1}^\ell b_j \neq 1$. Then there exists a unique homomorphism mapping $\Psi(w)$: $\mathcal{M}_1 \longrightarrow \mathcal{M}_2$ satisfying the functional equation (12) and (40) for all $w \in \mathcal{M}_1$ and all $\eta > 0$.

Theorem 4.6 If $\mathcal{W} : \mathcal{M}_1 \to \mathcal{M}_2$ is a function satisfying the inequalities (13) and (54). Assume λ_d is defined in (41) and $W : \mathcal{M}_1^\ell \to [0,\infty)$ be a function with the condition (42) for all $w_1, w_2, \cdots, w_\ell \in \mathcal{M}_1$ and all $\eta > 0$. If there exists $L = L(d)$ such that the functions (43) has the property (44) for all $w \in \mathcal{M}_1$ and all $\eta > 0$. Then there exists a unique homomorphism mapping $\Psi(w) : \mathcal{M}_1 \to \mathcal{M}_2$ satisfying the functional equation (12) and (45) for all $w \in \mathcal{M}_1$ and all $\eta > 0$.

Corollary 4.7 If $\mathcal{W} : \mathcal{M}_1 \to \mathcal{M}_2$ is a function satisfying the inequalities (39) and (56) for all $w_1, w_2.\cdots, w_\ell \in \mathcal{M}_1$ and all $\eta > 0$, with positive numbers $a > 0$ and $b \neq 1, \frac{1}{\ell}$. Then there exists a unique homomorphism mapping $\Psi(w) : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ satisfying the functional equation (12) and (40) for all $w \in \mathcal{M}_1$ and all $\eta > 0$.

Derivation Stability Result: Hyers and Radus Method

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Theorem 4.8 If $\mathcal{W} : \mathcal{M}_1 \to \mathcal{M}_2$ is a function satisfying the inequalities (13) and

$$\left.\begin{aligned} \mu\left(\Psi\left(\prod_j^{\ell} w_j\right) - \prod_j^{\ell} w_i \Psi(w_j), \eta\right) \geq \mu'\left(W\left(w_1, w_2, \cdots, w_\ell\right), \eta\right) \\ \nu\left(\Psi\left(\prod_j^{\ell} w_j\right) - \prod_j^{\ell} w_i \Psi(w_j), \eta\right) \leq \nu'\left(W\left(w_1, w_2, \cdots, w_\ell\right), \eta\right) \end{aligned}\right\} \quad (57)$$

where $\mathcal{W} : \mathcal{M}_1^{\ell} \longrightarrow (0,1]$ with the conditions (14), (15) and

$$\left.\begin{aligned} \lim_{\alpha\to\infty} \mu'\left(W\left(\gamma^{\ell\alpha\beta} w_1, \gamma^{\ell\alpha\beta} w_2, \cdots, \gamma^{\ell\alpha\beta} w_\ell\right), \gamma^{\ell\alpha\beta}\eta\right) = 1 \\ \lim_{\alpha\to\infty} \nu'\left(W\left(\gamma^{\ell\alpha\beta} w_1, \gamma^{\ell\alpha\beta} w_2, \cdots, \gamma^{\ell\alpha\beta} w_\ell\right), \gamma^{\ell\alpha\beta}\eta\right) = 0 \end{aligned}\right\} \quad (58)$$

for all $w_1, w_2, \cdots, w_\ell \in \mathcal{M}_1$ and all $\eta > 0$ with (16) holds. Then there exists a unique derivation mapping $\Psi(w) : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ given in (17) and satisfying the functional equation (12) and (18) for all $w \in \mathcal{M}_1$ and all $\eta > 0$.

Proof. In view of Theorem 3.1, there exists a unique additive mapping $\Psi(w) : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ given in (17) and satisfying the functional equation (12) and (18) for all $w \in \mathcal{M}_1$ and all $\eta > 0$. It follows from (57) and (58), one can obtain

$$\left.\begin{aligned} \mu\left(\Psi\left(\prod_j^{\ell} w_j\right) - \prod_j^{\ell} w_i \Psi(w_j), \eta\right) &= \mu\left(\frac{1}{\gamma^{\ell\alpha}}\left(\mathcal{W}\left(\prod_j^{\ell} \gamma^{\ell\alpha} w_j\right) - \prod_j^{\ell} \gamma^{\alpha} w_j \mathcal{W}(\gamma^{\ell\alpha} w_j)\right), \gamma^{\ell\alpha}\eta\right) \\ &\geq \mu'\left(W\left(\gamma^{\ell\alpha} w_1, \gamma^{\ell\alpha} w_2, \cdots, \gamma^{\ell\alpha} w_\ell\right) \gamma^{\ell\alpha}\eta\right) \\ &\rightarrow \quad 1 \quad \text{as} \quad \alpha \rightarrow \infty \\ \nu\left(\Psi\left(\prod_j^{\ell} w_j\right) - \prod_j^{\ell} w_i \Psi(w_j), \eta\right) &= \nu\left(\frac{1}{\gamma^{\ell\alpha}}\left(\mathcal{W}\left(\prod_j^{\ell} \gamma^{\ell\alpha} w_j\right) - \prod_j^{\ell} \gamma^{\alpha} w_j \mathcal{W}(\gamma^{\ell\alpha} w_j)\right), \gamma^{\ell\alpha}\eta\right) \\ &\leq \nu'\left(W\left(\gamma^{\ell\alpha} w_1, \gamma^{\ell\alpha} w_2, \cdots, \gamma^{\ell\alpha} w_\ell\right) \gamma^{\ell\alpha}\eta\right) \\ &\rightarrow \quad 0 \quad \text{as} \quad \alpha \rightarrow \infty \end{aligned}\right\}$$

for all $w \in \mathcal{M}_1$ and all $\eta > 0$. Thus $\Psi(w)$ is a unique derivation mapping.

Corollary 4.9 If $\mathcal{W} : \mathcal{M}_1 \to \mathcal{M}_2$ is a function satisfying the inequalities (39) and

$$\left.\begin{array}{l} \mu\left(\Psi\left(\prod_j^\ell w_j\right) - \prod_j^\ell w_i \Psi(w_j), \eta\right) \geq \left\{\begin{array}{l} \mu'(a,\eta), \\ \mu'\left(a\sum_{j=1}^{\ell} |w_j|^b, \eta\right), \\ \mu'\left(\sum_{j=1}^{\ell} a_j |w_j|^{b_j}, \eta\right), \\ \mu'\left(a\prod_{j=1}^{\ell} |w_j|^b, \eta\right), \\ \mu'\left(\prod_{j=1}^{\ell} a_j |w_j|^{b_j}, \eta\right), \end{array}\right. \\ \nu\left(\Psi\left(\prod_j^\ell w_j\right) - \prod_j^\ell w_i \Psi(w_j), \eta\right) \leq \left\{\begin{array}{l} \nu'(a,\eta), \\ \nu'\left(a\sum_{j=1}^{\ell} |w_j|^b, \eta\right), \\ \nu'\left(\sum_{j=1}^{\ell} a_j |w_j|^{b_j}, \eta\right), \\ \nu'\left(a\prod_{j=1}^{\ell} |w_j|^b, \eta\right), \\ \nu'\left(\prod_{j=1}^{\ell} a_j |w_j|^{b_j}, \eta\right), \end{array}\right. \end{array}\right\} \quad (59)$$

for all $w_1, w_2. \cdots, w_\ell \in \mathcal{M}_1$ and all $\eta > 0$, with positive numbers $a > 0$ and $b, b_j, \ell b, \sum_{j=1}^{\ell} b_j \neq 1$. Then there exists a unique derivation mapping $\Psi(w) : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ satisfying the functional equation (12) and (40) for all $w \in \mathcal{M}_1$ and all $\eta > 0$.

Theorem 4.10 If $\mathcal{W} : \mathcal{M}_1 \to \mathcal{M}_2$ is a function satisfying the inequalities (13) and (57). Assume λ_d is defined in (41) and $W : \mathcal{M}_1^\ell \to [0, \infty)$ be a function with the condition (42) for all $w_1, w_2, \cdots, w_\ell \in \mathcal{M}_1$ and all $\eta > 0$. If there exists $L = L(d)$ such that the functions (43) has the property (44) for all $w \in \mathcal{M}_1$ and all $\eta > 0$. Then there exists a unique derivation mapping $\Psi(w) : \mathcal{M}_1 \to \mathcal{M}_2$ satisfying the functional equation (12) and (45) for all $w \in \mathcal{M}_1$ and all $\eta > 0$.

Corollary 4.11 If $\mathcal{W} : \mathcal{M}_1 \to \mathcal{M}_2$ is a function satisfying the inequalities (39) and (59) for all $w_1, w_2. \cdots, w_\ell \in \mathcal{M}_1$ and all $\eta > 0$, with positive numbers $a > 0$ and $b \neq 1, \frac{1}{\ell}$. Then there exists a unique derivation mapping $\Psi(w) : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ satisfying the functional equation (12) and (40) for all $w \in \mathcal{M}_1$ and all $\eta > 0$.

Stability In Intuitionistic Fuzzy Banach Space and Algebra: Another Substitution

Hyers Method of (12)

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Theorem 5.1 If $\mathcal{W} : \mathcal{N}_1 \to \mathcal{N}_2$ is a function satisfying the inequalities

$$\left.\begin{array}{l} \mu\left(\mathcal{W}\left(\sum_{j=1}^{\ell}\ j\ w_j\right) - \sum_{j=1}^{\ell}\left(j\ \mathcal{W}(w_j)\right), \eta\ \right) \geq \mu'\left(W\left(w_1, w_2, \cdots, w_\ell\right), \eta\ \right) \\ \nu\left(\mathcal{W}\left(\sum_{j=1}^{\ell}\ j\ w_j\right) - \sum_{j=1}^{\ell}\left(j\ \mathcal{W}(w_j)\right), \eta\ \right) \leq \nu'\left(W\left(w_1, w_2, \cdots, w_\ell\right), \eta\ \right) \end{array}\right\} \quad (60)$$

where $W : \mathcal{N}_1^\ell \longrightarrow (0, 1]$ with the conditions

$$\left.\begin{array}{l} \mu'\left(W\left(\epsilon^{\alpha\beta}w_1, \epsilon^{\alpha\beta}w_2, \cdots, \epsilon^{\alpha\beta}w_\ell\right), \eta\ \right) \geq \mu'\left(\omega^{\alpha\beta}W\left(w_1, w_2, \cdots, w_\ell\right), \eta\right) \\ \nu'\left(W\left(\epsilon^{\alpha\beta}w_1, \epsilon^{\alpha\beta}w_2, \cdots, \epsilon^{\alpha\beta}w_\ell\right), \eta\ \right) \leq \nu'\left(\omega^{\alpha\beta}W\left(w_1, w_2, \cdots, w_\ell\right), \eta\right) \end{array}\right\} \quad (61)$$

and

$$\left.\begin{array}{l} \lim_{\alpha\to\infty} \mu'\left(W\left(\epsilon^{\alpha\beta}w_1, \epsilon^{\alpha\beta}w_2, \cdots, \epsilon^{\alpha\beta}w_\ell\right), \epsilon^{\alpha\beta}\eta\right) = 1 \\ \lim_{\alpha\to\infty} \nu'\left(W\left(\epsilon^{\alpha\beta}w_1, \epsilon^{\alpha\beta}w_2, \cdots, \epsilon^{\alpha\beta}w_\ell\right), \epsilon^{\alpha\beta}\eta\right) = 0 \end{array}\right\} \quad (62)$$

for all $w_1, w_2, \cdots, w_\ell \in \mathcal{N}_1$ and all $\eta > 0$ with

$$\beta = \pm 1 \qquad \text{and} \qquad 0 < \left(\frac{w}{\epsilon}\right)^\beta < 1. \quad (63)$$

Then there exists a unique additive mapping $\Psi(w) : \mathcal{N}_1 \longrightarrow \mathcal{N}_2$ given by

$$\left.\begin{array}{l} \lim_{\alpha\to\infty} \mu\left(\frac{\mathcal{W}(\epsilon^\alpha w)}{\epsilon^\alpha} - \Psi(w), \eta\ \right) = 1, \\ \lim_{\alpha\to\infty} \nu\left(\frac{\mathcal{W}(\epsilon^\alpha w)}{\epsilon^\alpha} - \Psi(w), \eta\ \right) = 0 \end{array}\right\} \quad (64)$$

and satisfying the functional equation (12) and

$$\left.\begin{array}{l} \mu\left(\Psi(w) - \mathcal{W}(w), \eta\ \right) \geq \mu'\left(W\left(w, 0, \cdots, 0, w\right), |\epsilon - \omega|\eta\right) \\ \nu\left(\Psi(w) - \mathcal{W}(w), \eta\ \right) \leq \nu'\left(W\left(w, 0, \cdots, 0, w\right), |\epsilon - \omega|\eta\right) \end{array}\right\} \quad (65)$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$.

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Proof. **Case (i):** Rechanging $w_2 = \cdots = w_{\ell-1} = 0; w_1 = w_\ell = w$ in (60), one can arrive

$$\left.\begin{array}{l} \mu\left(\mathcal{W}\left((1+\ell)\ \ w\right) - (1+\ell)\,\mathcal{W}(w), \eta\ \right) \geq \mu'\left(W\left(w, 0, \cdots, 0, w\right), \eta\ \right) \\ \nu\left(\mathcal{W}\left((1+\ell)\ \ w\right) - (1+\ell)\,\mathcal{W}(w), \eta\ \right) \leq \nu'\left(W\left(w, 0, \cdots, 0, w\right), \eta\ \right) \end{array}\right\} \tag{66}$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$. Define

$$1 + \ell = \epsilon$$

in (66), one can have

$$\left.\begin{array}{l} \mu\left(\mathcal{W}\left(\epsilon\ w\right) - \epsilon\mathcal{W}(w), \eta\ \right) \geq \mu'\left(W\left(w, w, \cdots, w\right), \eta\ \right) \\ \nu\left(\mathcal{W}\left(\epsilon\ w\right) - \epsilon\mathcal{W}(w), \eta\ \right) \leq \nu'\left(W\left(w, w, \cdots, w\right), \eta\ \right) \end{array}\right\} \tag{67}$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$. The rest of the proof is similar to that of Theorem 3.1.

The resulting corollary is an immediate consequence of Theorem 5.1, regarding the stability of the functional equation (12).

Corollary 5.2 If $\mathcal{W} : \mathcal{N}_1 \to \mathcal{N}_2$ is a function satisfying the inequalities

$$\left.\begin{array}{l} \mu\left(\mathcal{W}\left(\sum_{j=1}^{\ell}\ j\ w_j\right) - \sum_{j=1}^{\ell}\left(j\ \mathcal{W}(w_j)\right), \eta\ \right) \geq \left\{\begin{array}{l} \mu'\left(a, \eta\right), \\ \mu'\left(a\left(|w_1|^b + |w_\ell|^b\right), \eta\right), \\ \mu'\left(a_1|w_1|^{b_1} + a_\ell|w_\ell|^{b_\ell}, \eta\right), \end{array}\right. \\ \nu\left(\mathcal{W}\left(\sum_{j=1}^{\ell}\ j\ w_j\right) - \sum_{j=1}^{\ell}\left(j\ \mathcal{W}(w_j)\right), \eta\ \right) \leq \left\{\begin{array}{l} \nu'\left(a, \eta\right), \\ \nu'\left(a\left(|w_1|^b + |w_\ell|^b\right), \eta\right), \\ \nu'\left(a_1|w_1|^{b_1} + a_\ell|w_\ell|^{b_\ell}, \eta\right), \end{array}\right. \end{array}\right\} \tag{68}$$

for all $w_1, w_2. \cdots, w_\ell \in \mathcal{N}_1$ and all $\eta > 0$, with positive numbers $a > 0$ and $b, b_1, b_\ell \neq 1$. Then there exists a unique additive mapping $\Psi(w) : \mathcal{N}_1 \longrightarrow \mathcal{N}_2$ satisfying the

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functional equation (12) and

$$\left.\begin{array}{l}\mu\left(\mathcal{W}(w)-\Psi(w),\eta\ \right)\geq\begin{cases}\mu'\left(a,|\epsilon-1|\eta\right),\\ \mu'\left(2a\ |w|^{b},|\epsilon-\epsilon^{b}|\eta\right),\\ \mu'\left(a_1\ |w|^{b_1}+a_\ell\ |w|^{b_\ell},\eta\ \left(|\epsilon-\epsilon^{b_1}|+|\epsilon-\epsilon^{b_\ell}|\right)\right),\end{cases}\\ \nu\left(\mathcal{W}(w)-\Psi(w),\eta\ \right)\leq\begin{cases}\nu'\left(a,|\epsilon-1|\eta\right),\\ \nu'\left(2a\ |w|^{b},|\epsilon-\epsilon^{b}|\eta\right),\\ \nu'\left(a_1\ |w|^{b_1}+a_\ell\ |w|^{b_\ell},\eta\ \left(|\epsilon-\epsilon^{b_1}|+|\epsilon-\epsilon^{b_\ell}|\right)\right),\end{cases}\end{array}\right\} \tag{69}$$

for all $w\in\mathcal{N}_1$ and all $\eta>0$.

Radus Method of (12)

Theorem 5.3 If $\mathcal{W}:\mathcal{N}_1\rightarrow\mathcal{N}_2$ is a function satisfying the inequality (60). Assume

$$\lambda_d=\begin{cases}\epsilon & d=0;\\ \frac{1}{\epsilon} & d=1.\end{cases} \tag{70}$$

and $W:\mathcal{N}_1^{\ell}\rightarrow[0,\infty)$ be a function with the condition

$$\left.\begin{array}{l}\lim\limits_{\alpha\rightarrow\infty}\mu'\left(W\left(\lambda_d^{\alpha}w_1,\lambda_d^{\alpha}w_2,\cdots,\lambda_d^{\alpha}w_\ell\right),\lambda_d^{\alpha}\eta\right)=1\\ \lim\limits_{\alpha\rightarrow\infty}\nu'\left(W\left(\lambda_d^{\alpha}w_1,\lambda_d^{\alpha}w_2,\cdots,\lambda_d^{\alpha}w_\ell\right),\lambda_d^{\alpha}\eta\right)=0\end{array}\right\} \tag{71}$$

for all $w_1,w_2,\cdots,w_\ell\in\mathcal{N}_1$ and all $\eta>0$. If there exists $L=L(d)$ such that the functions

$$\left.\begin{array}{l}\mu'\left(W(w,0,\cdots,0,w),\eta\right)=\mu'\left(W\left(\frac{w}{\epsilon},0,\cdots,0,\frac{w}{\epsilon}\right),\eta\right);\\ \nu'\left(W(w,0,\cdots,0,w),\eta\right)=\nu'\left(W\left(\frac{w}{\epsilon},0,\cdots,0,\frac{w}{\epsilon}\right),\eta\right)\end{array}\right\} \tag{72}$$

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has the property

$$\left.\begin{array}{l} \mu'\left(\frac{1}{\lambda_d}W\left(\lambda_d w, 0, \cdots, 0, \lambda_d w\right), \eta\right) = \mu'\left(L\ W(w, 0, \cdots, 0, w), \eta\right) \\ \nu'\left(\frac{1}{\lambda_d}W\left(\lambda_d w, 0, \cdots, 0, \lambda_d w\right), \eta\right) = \nu'\left(L\ W(w, 0, \cdots, 0, w), \eta\right) \end{array}\right\} \tag{73}$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$. Then there exists a unique additive mapping $\Psi(w) : \mathcal{N}_1 \to \mathcal{N}_2$ satisfying the functional equation (12) and

$$\left.\begin{array}{l} \mu\left(\mathcal{W}(w) - \Psi(w), \eta\right) \geq \mu'\left(\frac{L^{1-d}}{1-L}W(w, 0, \cdots, 0, w), \eta\right) \\ \nu\left(\mathcal{W}(w) - \Psi(w), \eta\right) \leq \nu'\left(\frac{L^{1-d}}{1-L}W(w, 0, \cdots, 0, w), \eta\right) \end{array}\right\} \tag{74}$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$.

Corollary 5.4 If $\mathcal{W} : \mathcal{N}_1 \to \mathcal{N}_2$ is a function satisfying the inequalities

$$\left.\begin{array}{l} \mu\left(\mathcal{W}\left(\sum_{j=1}^{\ell}\ j\ w_j\right) - \sum_{j=1}^{\ell}\left(j\ \mathcal{W}(w_j)\right), \eta\ \right) \geq \left\{\begin{array}{l} \mu'\left(a, \eta\right), \\ \mu'\left(a \sum_{j=1}^{\ell} |w_j|^b, \eta\right), \end{array}\right. \\ \nu\left(\mathcal{W}\left(\sum_{j=1}^{\ell}\ j\ w_j\right) - \sum_{j=1}^{\ell}\left(j\ \mathcal{W}(w_j)\right), \eta\ \right) \leq \left\{\begin{array}{l} \nu'\left(a, \eta\right), \\ \nu'\left(a \sum_{j=1}^{\ell} |w_j|^b, \eta\right), \end{array}\right. \end{array}\right\} \tag{75}$$

for all $w_1, w_2. \cdots, w_\ell \in \mathcal{N}_1$ and all $\eta > 0$, with positive numbers $a > 0$ and $b \neq 1$. Then there exists a unique additive mapping $\Psi(w) : \mathcal{N}_1 \longrightarrow \mathcal{N}_2$ satisfying the functional equation (12) and

$$\left.\begin{array}{r} \mu\left(W(w) - \Psi(w), \eta\ \right) \geq \left\{\begin{array}{l} \mu'\left(a, |\epsilon - 1|\eta\right), \\ \mu'\left(2a\ |w|^b, |\epsilon - \epsilon^b|\eta\right), \end{array}\right. \\ \nu\left(W(w) - \Psi(w), \eta\ \right) \leq \left\{\begin{array}{l} \nu'\left(a, |\epsilon - 1|\eta\right), \\ \nu'\left(2a\ |w|^b, |\epsilon - \epsilon^b|\eta\right), \end{array}\right. \end{array}\right\} \tag{76}$$

for all $w \in \mathcal{N}_1$ and all $\eta > 0$.

Homomorphism Stability Result: Hyers and Radus Method

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Theorem 5.5 If $\mathcal{W} : \mathcal{M}_1 \to \mathcal{M}_2$ is a function satisfying the inequalities (60) and

$$\left.\begin{aligned} \mu\left(\Psi\left(w_1 w_\ell\right) - \Psi(w_1)\Psi(w_\ell), \eta\ \right) \geq \mu'\left(W\left(w_1, 0, \cdots, 0, w_\ell\right), \eta\ \right) \\ \nu\left(\Psi\left(w_1 w_\ell\right) - \Psi(w_1)\Psi(w_\ell), \eta\ \right) \leq \nu'\left(W\left(w_1, 0, \cdots, 0, w_\ell\right), \eta\ \right) \end{aligned}\right\} \tag{77}$$

where $\mathcal{W} : \mathcal{M}_1^\ell \longrightarrow (0, 1]$ with the conditions (61), (62) and

$$\left.\begin{aligned} \lim_{\alpha\to\infty} \mu'\left(W\left(\epsilon^{\alpha\beta} w_1, \epsilon^{\alpha\beta} w_2, \cdots, \epsilon^{\alpha\beta} w_\ell\right), \epsilon^{\ell\alpha\beta}\eta\right) = 1 \\ \lim_{\alpha\to\infty} \nu'\left(W\left(\epsilon^{\alpha\beta} w_1, \epsilon^{\alpha\beta} w_2, \cdots, \epsilon^{\alpha\beta} w_\ell\right), \epsilon^{\ell\alpha\beta}\eta\right) = 0 \end{aligned}\right\} \tag{78}$$

for all $w_1, w_2, \cdots, w_\ell \in \mathcal{M}_1$ and all $\eta > 0$ with (16) holds. Then there exists a unique homomorphism mapping $\Psi(w) : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ given in (64) and satisfying the functional equation (12) and (65) for all $w \in \mathcal{M}_1$ and all $\eta > 0$.

Corollary 5.6 If $\mathcal{W} : \mathcal{M}_1 \to \mathcal{M}_2$ is a function satisfying the inequalities (68) and

$$\left.\begin{aligned} \mu\left(\Psi\left(w_1 w_\ell\right) - \Psi(w_1)\Psi(w_\ell), \eta\ \right) \geq \begin{cases} \mu'\left(a, \eta\right), \\ \mu'\left(a\left(|w_1|^b + |w_\ell|^b\right), \eta\right), \\ \mu'\left(a_1|w_1|^{b_1} + a_\ell|w_\ell|^{b_\ell}, \eta\right), \end{cases} \\ \nu\left(\Psi\left(w_1 w_\ell\right) - \Psi(w_1)\Psi(w_\ell), \eta\ \right) \leq \begin{cases} \nu'\left(a, \eta\right), \\ \nu'\left(a\left(|w_1|^b + |w_\ell|^b\right), \eta\right), \\ \nu'\left(a_1|w_1|^{b_1} + a_\ell|w_\ell|^{b_\ell}, \eta\right), \end{cases} \end{aligned}\right\} \tag{79}$$

for all $w_1, w_2. \cdots, w_\ell \in \mathcal{M}_1$ and all $\eta > 0$, with positive numbers $a > 0$ and $b, b_1, b_\ell \neq 1$. Then there exists a unique homomorphism mapping $\epsilon(w) : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ satisfying the functional equation (12) and (69) for all $w \in \mathcal{M}_1$ and all $\eta > 0$.

Theorem 5.7 If $\mathcal{W} : \mathcal{M}_1 \to \mathcal{M}_2$ is a function satisfying the inequalities (60) and (77). Assume λ_d is defined in (70) and $W : \mathcal{M}_1^\ell \to [0, \infty)$ be a function with the condition (71) for all $w_1, w_2, \cdots, w_\ell \in \mathcal{M}_1$ and all $\eta > 0$. If there exists $L = L(d)$ such that the functions (72) has the property (73) for all $w \in \mathcal{M}_1$ and all $\eta > 0$. Then there exists a unique homomorphism mapping $\Psi(w) : \mathcal{M}_1 \to \mathcal{M}_2$ satisfying the functional equation (12) and (74) for all $w \in \mathcal{M}_1$ and all $\eta > 0$.

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Corollary 5.8 If $\mathcal{W} : \mathcal{M}_1 \to \mathcal{M}_2$ is a function satisfying the inequalities (68)) and (79) for all $w_1, w_2. \cdots, w_\ell \in \mathcal{M}_1$ and all $\eta > 0$, with positive numbers $a > 0$ and $b, b_1, b_\ell \neq 1$. Then there exists a unique homomorphism mapping $\Psi(w) : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ satisfying the functional equation (12) and (69) for all $w \in \mathcal{M}_1$ and all $\eta > 0$.

Derivation Stability Result: Hyers and Radus Method

Theorem 5.9 If $\mathcal{W} : \mathcal{M}_1 \to \mathcal{M}_2$ is a function satisfying the inequalities (60) and

$$\left.\begin{aligned} \mu\left(\Psi\left(w_1 w_\ell\right) - w_1\Psi(w_\ell) - w_\ell\Psi(w_1), \eta\ \right) \geq \mu'\left(W\left(w_1, 0, \cdots, 0, w_\ell\right), \eta\ \right) \\ \nu\left(\Psi\left(w_1 w_\ell\right) - w_1\Psi(w_\ell) - w_\ell\Psi(w_1), \eta\ \right) \leq \nu'\left(W\left(w_1, 0, \cdots, 0, w_\ell\right), \eta\ \right) \end{aligned}\right\} \tag{80}$$

where $\mathcal{W} : \mathcal{M}_1^\ell \longrightarrow (0, 1]$ with the conditions (61), (62) and

$$\left.\begin{aligned} \lim_{\alpha\to\infty} \mu'\left(W\left(\epsilon^{\alpha\beta} w_1, \epsilon^{\alpha\beta} w_2, \cdots, \epsilon^{\alpha\beta} w_\ell\right), \epsilon^{\ell\alpha\beta}\eta\right) = 1 \\ \lim_{\alpha\to\infty} \nu'\left(W\left(\epsilon^{\alpha\beta} w_1, \epsilon^{\alpha\beta} w_2, \cdots, \epsilon^{\alpha\beta} w_\ell\right), \epsilon^{\ell\alpha\beta}\eta\right) = 0 \end{aligned}\right\} \tag{81}$$

for all $w_1, w_2, \cdots, w_\ell \in \mathcal{M}_1$ and all $\eta > 0$ with (16) holds. Then there exists a unique derivation mapping $\Psi(w) : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ given in (64) and satisfying the functional equation (12) and (65) for all $w \in \mathcal{M}_1$ and all $\eta > 0$.

Corollary 5.10 If $\mathcal{W} : \mathcal{M}_1 \to \mathcal{M}_2$ is a function satisfying the inequalities (68) and

$$\left.\begin{aligned} \mu\left(\Psi\left(w_1 w_\ell\right) - w_1\Psi(w_\ell) - w_\ell\Psi(w_1), \eta\ \right) \geq \begin{cases} \mu'\left(a, \eta\right), \\ \mu'\left(a\left(|w_1|^b + |w_\ell|^b\right), \eta\right), \\ \mu'\left(a_1|w_1|^{b_1} + a_\ell|w_\ell|^{b_\ell}, \eta\right), \end{cases} \\ \nu\left(\Psi\left(w_1 w_\ell\right) - w_1\Psi(w_\ell) - w_\ell\Psi(w_1), \eta\ \right) \leq \begin{cases} \nu'\left(a, \eta\right), \\ \nu'\left(a\left(|w_1|^b + |w_\ell|^b\right), \eta\right), \\ \nu'\left(a_1|w_1|^{b_1} + a_\ell|w_\ell|^{b_\ell}, \eta\right), \end{cases} \end{aligned}\right\} \tag{82}$$

for all $w_1, w_2. \cdots, w_\ell \in \mathcal{M}_1$ and all $\eta > 0$, with positive numbers $a > 0$ and $b, b_1, b_\ell \neq 1$Then there exists a unique derivation mapping $\Psi(w) : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ satisfying the functional equation (12) and (69) for all $w \in \mathcal{M}_1$ and all $\eta > 0$.

Theorem 5.11 If $\mathcal{W} : \mathcal{M}_1 \to \mathcal{M}_2$ is a function satisfying the inequalities (60) and

(80). Assume λ_d is defined in (70) and $W : \mathcal{M}_1^\ell \to [0,\infty)$ be a function with the condition (71) for all $w_1, w_2, \cdots, w_\ell \in \mathcal{M}_1$ and all $\eta > 0$. If there exists $L = L(d)$ such that the functions (72) has the property (73) for all $w \in \mathcal{M}_1$ and all $\eta > 0$. Then there exists a unique derivation mapping $\Psi(w) : \mathcal{M}_1 \to \mathcal{M}_2$ satisfying the functional equation (12) and (74) for all $w \in \mathcal{M}_1$ and all $\eta > 0$.

Corollary 5.12 If $\mathcal{W} : \mathcal{M}_1 \to \mathcal{M}_2$ is a function satisfying the inequalities (68) and (82) for all $w_1, w_2. \cdots, w_\ell \in \mathcal{M}_1$ and all $\eta > 0$, with positive numbers $a > 0$ and $b \neq 1, \frac{1}{\ell}$. Then there exists a unique derivation mapping $\Psi(w) : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ satisfying the functional equation (12) and (69) for all $w \in \mathcal{M}_1$ and all $\eta > 0$.

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